Finding Strategies to Solve a 4x4x3 3D Domineering Game

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THESIS

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1.0 – Introduction to Combinatorial Games & Domineering

In this chapter, we will introduce several topics to help the reader understand the material covered in this thesis.

1.1 – Combinatorial Games

Combinatorial game theory is the study of combinatorial games and differs from the traditional analysis of game theory, which reviews economic, zero-sum games. In zero-sum games, the gains of one player are offset by the losses of the other player, which equals the sum of zero. In these games, the outcome of how each player wins or loses is determined by a payoff matrix, and neither player will know exactly what the other player will do since they both have to move simultaneously. The following section explains what a combinatorial game is and how it differs from economic games.

What is a combinatorial game?

A combinatorial game is a game where two players (often called L for left player and R for right player) play in alternate turns until one player is unable to make any legal moves. Usually, the player who cannot make any more legal moves loses, but there are variations to the combinatorial game called misére where the player who cannot make any more legal moves wins. There are no ties in a combinatorial game—there must be one winner and one loser. Nothing about the combinatorial game is hidden from either player as both players know all the details of the game's state at all times (this is referred to as perfect information). A combinatorial game has no element of chance like a dice roll or a card deal. There are also no cycles in a combinatorial game, meaning that a position in a combinatorial game is not repeated in a later turn. For this thesis, we will only consider finite combinatorial games, which end after a finite sequence of moves.

If a game violates any of the above criteria, then it is not a combinatorial game. For example, checkers is not considered a combinatorial game because it can have cycles, meaning that positions can be repeated. Chess also cannot be classified as a combinatorial game because the game can end in a draw. Zero-sum games, which we mentioned in the beginning of this chapter, are not combinatorial games because moves by both players are performed simultaneously instead of in alternate turns.

Now that we have provided examples of what does not constitute a combinatorial game, we will show a game that does fulfill the requirements of a combinatorial game, Domineering.

What is Domineering?

Domineering is a two-player combinatorial game that is played on a grid board of any size (ex. 3x3, 4x6, 10x4, etc.), although it is usually square to be fair to both players. Each player can place a domino piece that covers two adjacent spaces on the board. Both players take turns placing a domino on the board (the domino cannot overlap other domino pieces) until one player is unable to place any more pieces, which results in a loss for that player. One player, referred to as left player or L, can only place pieces vertically on the board. The other player, referred to as right player or R, can only place pieces horizontally on the board. Diagrams 1-1a to 1-1c show a brief example of a Domineering game. In this and subsequent examples of Domineering games, moves made in the current turn will appear black, and moves made in previous turns will appear gray.

Example of a 3x3 Domineering Game

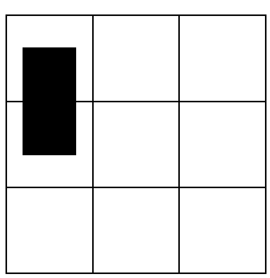


Diagram 1-1a
L makes his first move.

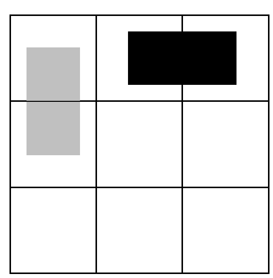


Diagram 1-1b
R then makes his first move.

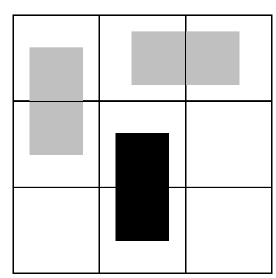


Diagram 1-1c
L then makes his second move.

In this 3x3 Domineering game, the first player (L) starts by placing a piece vertically on the upper-left corner of the board in Diagram 1-1a. The second player (R) places a piece horizontally on the upper-right corner of the board in Diagram 1-1b. L places his second piece vertically on the center of the board in Diagram 1-1c. This move prevents R from placing any more horizontal pieces on the board, which results in a loss for R.

Domineering is classified as a partizan combinatorial game.

Impartial vs. Partizan Combinatorial Games

There are two types of combinatorial games: impartial and partizan. An impartial combinatorial game is one where, in any position, the moves available to be played by either player are identical. A partizan combinatorial game is a game where the moves both players make are different. To show the differences between impartial and partizan games, we will introduce another combinatorial game called Nim. Nim is played with heaps of counters, as demonstrated in the diagram below.

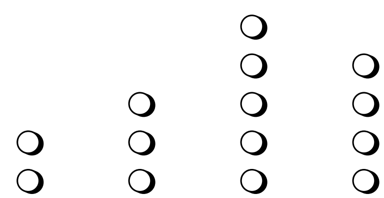


Diagram 1-2
An example of Nim

Each player takes turns removing any number of counters (but at least one) from one heap until there are no more counters (a player loses when there are no more counters to remove in his turn). Nim is an impartial game because both left and right players have the same moves: removing any number of counters from a heap. Domineering is a partizan game because both players have different moves: one player can only place pieces vertically on the board, while the other can only place pieces horizontally on the board.

It is important to note the distinction between impartial and partizan games because the number of outcome classes is smaller for impartial games.

What is an outcome class?

An outcome class describes which player in a combinatorial game is able to win regardless of what his opponent does (also known as “forcing a win”) provided that the player follows a specific strategy (also known as the winning strategy). It is derived from the “Fundamental Theorem of Combinatorial Games” introduced in *Lessons in Play* [ANW_07], which is defined as:

In two-player finite combinatorial games, exactly one player always has a winning strategy for a given turn order.

In *Lessons in Play*, the authors presented the proof of this theorem as follows: presume a finite combinatorial game G where the first player can force a win going first or the second player can force a win going second, but not both. Each of the first player’s moves goes to a position that, by induction, is either a win for the second player going first or the first player going second. If any of the first player’s moves belongs to the latter category, then the first player can force a win by choosing one of them. However, if all of the first player’s moves are in the former category, then the second player can force a win by using his winning strategy in the position resulting from any of the first player’s moves [ANW_07].

From the “Fundamental Theorem of Combinatorial Games”, we can define two outcome classes for impartial combinatorial games: $\mathcal{N}$, which means the first player (or next player) can force a win, and $\mathcal{P}$, which means the second player (or previous player) can force a win. The following diagram shows an example of an $\mathcal{N}$ game using Nim.

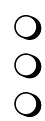


Diagram 1-3
A Nim game in outcome class $\mathcal{N}$

The Nim game in Diagram 1-3 is in $\mathcal{N}$ because the first player can always force a win by removing all the counters from the single heap. The following shows an example of a $\mathcal{P}$ game using Nim.

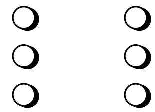


Diagram 1-4
A Nim game in outcome class $\mathcal{P}$

The Nim game in Diagram 1-4 is in $\mathcal{P}$ because the second player can force a win by keeping the two heap stacks even until one heap is completely removed by the first player (the second player can then remove what's left from the remaining heap to win). For example, if the first player were to remove one counter from one heap, the second player should remove one counter from the other heap. As long as the second player follows the winning strategy of keeping the heaps even, there is nothing that the first player can do to win.

When the Fundamental Theorem is applied to partizan combinatorial games, there are four outcome classes, which are defined in the following chart.

Outcome Class	When Left moves first	When Right moves first
$\mathcal{N}$	Left wins	Right wins
$\mathcal{P}$	Right wins	Left wins
$\mathcal{L}$	Left wins	Left wins
$\mathcal{R}$	Right wins	Right wins

Diagram 1-5
Outcome classes for partizan combinatorial games

In additional to $\mathcal{N}$ and $\mathcal{P}$, partizan games also have the outcome classes $\mathcal{L}$, which means that the left player can force a win regardless of turn order, and $\mathcal{R}$, which means that the right player can force a win regardless of turn order. Partizan combinatorial games have more outcome classes than impartial combinatorial games because the moves that left and right player make in partizan games are different. The following demonstrate examples of $\mathcal{L}$ and $\mathcal{R}$ games using Domineering.

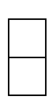


Diagram 1-6a
A Domineering game in outcome class $\mathcal{L}$

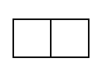


Diagram 1-6b
A Domineering game in outcome class $\mathcal{R}$

Diagram 1-6a shows a $\mathcal{L}$ Domineering game where left player can win regardless of turn order because right player is unable to move in that game. Likewise, Diagram 1-6b shows a $\mathcal{R}$ Domineering game where right player can win regardless of turn order because left player is unable to move in that game.

In the next section, we will introduce another important aspect of combinatorial games, their game tree.

Combinatorial Game Trees

A game tree is a directed graph whose nodes represent all the game's possible positions and whose edges represent all possible moves that can be made in the game. Typically, the positions on the game tree's even levels represent moves made by the first player, and the positions on the tree's odd levels represent moves made by the second player [Ros_03]. The following diagram shows an example of a game tree with the first player's moves marked with 'L', and the second player's moves marked with 'R'.

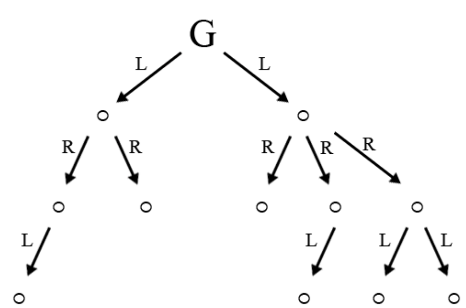


Diagram 1-7
An example of a game tree

However, the game tree of a combinatorial game is constructed differently, in part because of partizan combinatorial games. Recall that in a partizan combinatorial game, the moves left and right player make are not the same. Therefore, the children of a position in a combinatorial game tree must be partitioned into two sets, one for the left player and one for the right player. Moves that can be made at a position by the left player are drawn as the position's left children and are considered to be the position's left options. Moves that can be made at a position by the right player are drawn as the position's right children and are considered to be the position's right options. The following diagram shows an example of a game tree for a combinatorial game.

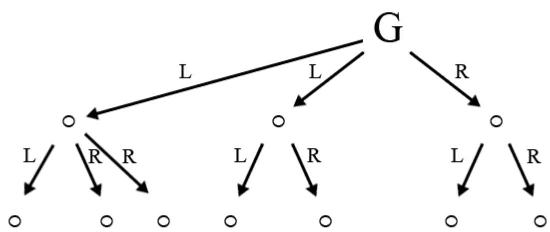


Diagram 1-8
An example of a combinatorial game tree

One could consider a combinatorial game tree as a combination of two game trees (with each tree having a different player going first). However, the reader may notice in Diagram 1-8 that several positions in the tree are the result of two consecutive left or right options, even though such moves are not possible in a game with alternating turns. There are two reasons for this. The first is that both children are needed to calculate the outcome class of a position (which we will demonstrate later in this chapter). The second is that a game tree can represent a summand in a sum of games. Thus, it is possible that one player can move consecutively on one summand game if the other player plays on another summand game.

The Value of a Combinatorial Game

Now that we have introduced the combinatorial game tree, we can use them to demonstrate the recursive definition of a combinatorial game. A combinatorial game, G, can be defined as $\{\mathcal{G}^L \mid \mathcal{G}^R\}$, where $\mathcal{G}^L$ represents the set of G's left options (which are also combinatorial games) and $\mathcal{G}^R$ represents the set of G's right options (also combinatorial games) [ANW_07]. The base case for this recursive definition is $\mathcal{G}^L = \varnothing$ and $\mathcal{G}^R = \varnothing$ (Ø represents the null set) because there exist combinatorial games where the left and right players will not be able to make any moves. From this recursive definition, J.L. Conway developed the simplest possible combinatorial games and defined values for each of them, which are demonstrated in Diagram 1-11 [Con_76]. We also listed the outcome classes of each game to show how Conway's simple games correlate to the outcome classes we defined earlier in this chapter.

{ | } = 0 $\mathcal{P}$ { 0 | } = 1 $\mathcal{L}$ { | 0 } = -1 $\mathcal{R}$ { 0 | 0 } = * $\mathcal{N}$

Diagram 1-9
Conway's simple combinatorial games and their outcome classes

The first game in Diagram 1-9 is defined as 0 and has no left or right options (meaning $\mathcal{G}^L$ and $\mathcal{G}^R$ are empty, and { Ø | Ø } can be rewritten as { | }). This game is in the $\mathcal{P}$ outcome class, a win for the second player. This is because whoever goes first automatically loses, as there are no moves available. Whenever you see the game 0 mentioned later in this chapter, remember that it is a combinatorial game where neither left nor right player can make a move.

The second game in Diagram 1-9 is defined as 1 and has 0 (the first game in Diagram 1-9) as its left option and the empty set as its right option. Since the right player cannot move in this game regardless of whether he goes first or second, the second game is a win for the left player, so it is in outcome class $\mathcal{L}$.

The third game in Diagram 1-9 is defined as -1 and has the empty set as its left option and 0 (the first game in Diagram 1-9) as its right option. Since the left player cannot move in this game regardless of whether he goes first or second, the third game is a win for the right player, so it is in outcome class $\mathcal{R}$.

The last game in Diagram 1-9 is defined as * and has 0 as its left and right options. It is in the $\mathcal{N}$ outcome class, a win for the first player, because after the first player moves, the second player cannot move and therefore loses.

Canonical Form of Combinatorial Games

An interesting property of combinatorial games that can make traversing a combinatorial game tree easier is that they can be reduced to a simpler game called the canonical form.

The canonical form of a game G is equivalent to G in the sense that it behaves the same as G in any sum of games—that is the outcome of the sum is preserved (see page 78 of [ANW_07]). This form can be reached by removing the dominated and reversible options of a game, which we will explain in this section.

To demonstrate what a dominated option is in a combinatorial game, we will use the following Domineering game in Diagram 1-10 as an example.

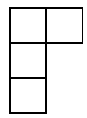


Diagram 1-10

If left player were to move first, he has two possible moves in this game, as demonstrated in Diagrams 1-11a and 1-11b as games H_1 and H_2.

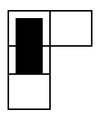


Diagram 1-11a
Game H_1

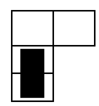


Diagram 1-11b
Game H_2

The move made in Diagram 1-11a (Game H_1) is great for the left player because the right player is unable to make a move, making it a win for the left player. The move made in Diagram 1-11b (Game H_2), on the other hand, is bad for the left player because he loses, as the right player covers the remaining spaces of the game with his next move. The second move is considered dominated by the first move because it is worse for the left player. Technically, $H_1 \geq H_2$, or L wins playing second on $H_1 - H_2$, which is equivalent to $H_1 - H_2 \in \mathcal{L} \cup \mathcal{P}$ [ANW_07]. When simplifying the game in Diagram 1-10 to its canonical form, the dominated option can be removed from the game tree.

To demonstrate what a reversible option is in a combinatorial game, we will introduce the combinatorial game, *up*, which is defined as the following [ANW_07]:

$$\uparrow = \{\ 0 \mid * \ \}$$

Diagram 1-12

This game is in the $\mathcal{L}$ outcome class because the left player can force a win regardless of whether he goes first or second. The left option is 0 (see the first game in Diagram 1-9), which is a game where the right player cannot move in his turn, resulting in the right player's loss. The right option is * (see the fourth game in Diagram 1-9), which is a win for the left player because * is an $\mathcal{N}$ game and the first player is the next player to move.

Next, we will construct a partial game tree for the game, $\uparrow$*.

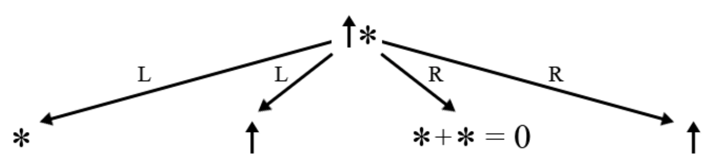


Diagram 1-13

$\uparrow$* also means $\uparrow$+*, so both players can either play on $\uparrow$ or * in their first turn. The positions from those possible moves are listed as the children in the Diagram 1-13 game tree. See the fourth game in Diagram 1-9 and the definition in Diagram 1-12 to review what happens when the left and right players play on * and $\uparrow$. The reader will notice that none of the child positions are listed as * + 0 or $\uparrow$ + 0, even though some of the results of playing on $\uparrow$ or * is 0 (although when right player plays on $\uparrow$, it is *). Since 0 is a game in which neither player can make any more moves, we can omit that game from the game tree if it is a summand with another game.

One of the possible positions from $\uparrow$* for the right player is * + * (if he played on $\uparrow$ in his first turn). We simplified * + * to 0 because when the first player moves on * + *, the remaining position is * + 0, or *. When the second player moves on *, the resulting position is 0.

The game's right option that leads to $\uparrow$ is dominated by the right option that leads to 0 because it is a worse move for right player. Recall that $\uparrow$ is in $\mathcal{L}$, or a win for the left player. Therefore, the right player is better off playing to the position that goes to 0, as that results in a win for the right player (since left player cannot move in the 0 game). We will remove the dominated option in subsequent game tree diagrams. The reader may notice in Diagram 1-13's left options that * is a worse move for left player than $\uparrow$, but because $\uparrow$-* is an $\mathcal{N}$ game, * cannot be removed as a dominated option.

A reversible option is defined as an option that an opposing player can immediately respond to such that it leaves the opposing player in as good or better position than before the reversible option is played. We will continue our $\uparrow$* game tree example to demonstrate this option.

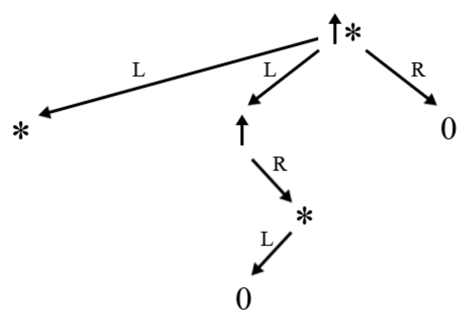


Diagram 1-14

Note that after right player moves on $\uparrow$, the position is *. After left plays on *, it is a loss for the right player because the right player cannot move in the 0 game. Therefore, the second level $\uparrow$ is a reversible option because its right option leaves the left player at a better position than $\uparrow$*. A reversible option and its subsequent option are collapsed into the game tree, which in this case is the second level $\uparrow$ and its right child, *.

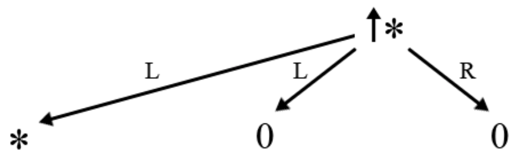


Diagram 1-15

There are no more dominated or reversible options, so we have reached the canonical form of $\uparrow *$, which is { 0, * | 0 }. For a proof that this simplified game is equivalent to the original game { *, $\uparrow$ | 0, $\uparrow$ }, see Chapter 4 of [ANW_07].

The First Solved Combinatorial Game

A combinatorial game is considered solved if one can find which player has a winning strategy for each turn order. The first combinatorial game to be solved was Nim, back in 1901 by Charles L. Bouton [Bou_02]. He presented a mathematical proof that stated the player's winning strategy for each turn order based on the number of counters and heaps in a Nim game. To present this proof, we first need to introduce *nim-sum*. Nim-sum is the binary digital sum of numbers that discards any carries from one digit to the next [ANW_07]. In other words, it is the "bit-wise" sum of the digits modulo 2. The nim-sum operand is $\oplus$ to distinguish it from +. The following shows an example of calculating $5 \oplus 6 \oplus 7$.

5	1	0	1
6	1	1	0
7	1	1	1
	1	0	0

Diagram 1-16
A demonstration to calculate $5 \oplus 6 \oplus 7$

The answer to the example in Diagram 1-16 is 100, which is binary for 4. For each column of binary digits, if there is an even number of 1s, then the column sum is 0; otherwise, the column sum is 1.

Earlier in this chapter, we showed the winning strategies for Nim when the number of heaps was 1 (that game is in outcome class $\mathcal{N}$) or 2 (that game is in outcome class $\mathcal{P}$). For all Nim games with heap size a, b, …, k, the games are in the outcome class $\mathcal{P}$ if $a \oplus b \oplus \ldots \oplus k = 0$; otherwise, they are in outcome class $\mathcal{N}$ [ANW_07]. We will demonstrate this using the following Nim game as an example.

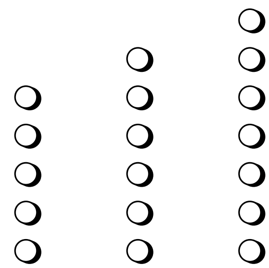


Diagram 1-17
A Nim game with heap sizes 5, 6, and 7

We already calculated the nim-sum of $5 \oplus 6 \oplus 7$ to be 4. Since the nim-sum is not 0, this game is in outcome class $\mathcal{N}$. The first player's winning strategy is to remove the correct number of counters from a heap so that the game's new nim-sum is 0. This can be achieved by nim-adding the game's current nim-sum (which in this case is 4) and the number of counters in one heap. If this new nim-sum is less than the current size of the heap, the first player can reduce the heap's size to create a 0 nim-sum game. Let's apply this strategy to the Diagram 1-17 game.

The Diagram 1-17 game has heap sizes 5, 6, and 7. If you nim-added those three values to the game's nim-sum (which is 4), you get 1, 2, and 3 respectively ($5 \oplus 4 = 1$, $6 \oplus 4 = 2$, and $7 \oplus 4 = 3$). Since $1 < 5$, $2 < 6$, and $3 < 7$, the first player can reduce any of the three heaps to those values to create a 0 nim-sum game, which is demonstrated in the following diagrams.

1	0	0	1
6	1	1	0
7	1	1	1
	0	0	0

Diagram 1-18a
$1 \oplus 6 \oplus 7 = 0$

5	1	0	1
2	0	1	0
7	1	1	1
	0	0	0

Diagram 1-18b
$5 \oplus 2 \oplus 7 = 0$

5	1	0	1
6	1	1	0
3	0	1	1
	0	0	0

Diagram 1-18c
$5 \oplus 6 \oplus 3 = 0$

In Diagram 1-18a, the first player reduced the heap of size 5 to 1. In Diagram 1-18b, the first player reduced the heap of size 6 to 2. In Diagram 1-18c, the first player reduced the heap of size 7 to 3. Note that the first player only needs to reduce one heap size to create a 0 nim-sum game.

No matter which and how many counters the second player removes in his next turn, the first player will always win as long as he follows the winning strategy of removing the correct number of counters to make the game's nim-sum 0.

A proof of this winning strategy can be found in pages 138-140 in [ANW_07]. For many other combinatorial games, a general solution is not common, but specific instances can be solved with a game tree analysis. In the next section, we will demonstrate how to use a game tree to solve a combinatorial game.

Solving a Combinatorial Game

One method to solve a combinatorial game is to construct its game tree. From that game tree, one can calculate the game's outcome class and determine whether a winning strategy for each turn order exists. This process is similar to the typical method of analyzing a game tree, which is to calculate the value of each tree's node and use a minimax strategy to pick the best moves for each player [Ros_03].

The minimax strategy is a strategy where the first player picks a position whose child has the maximum value and the second player picks a position whose child has the minimum value. The following diagram shows an example of a minimax tree.

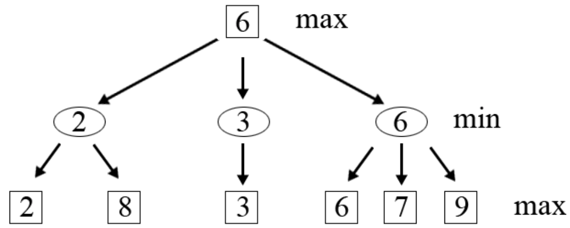


Diagram 1-19
An example of a minimax game tree

Note in Diagram 1-19 how the max player (whose nodes are squares) picked the maximum value from his available moves and the min player (whose nodes are ellipses) picked the minimum value from each of his available moves. Combinatorial games must have one winner and one loser, so the values can be restricted to 1 if it is a win for the first player and -1 if it is a win for the second player. The following chart demonstrates how those values correlate to the outcome classes we defined earlier in this chapter.

Outcome Class	When Left moves first	When Right moves first
$\mathcal{N}$	Left wins : 1	Right wins : 1
$\mathcal{P}$	Right wins : -1	Left wins : -1
$\mathcal{L}$	Left wins : 1	Left wins : -1
$\mathcal{R}$	Right wins : -1	Right wins : 1

Diagram 1-20

Once the outcome classes for each position of the game tree is calculated, the minimax strategy can be used to determine the player's best moves for each turn order in a combinatorial game. The next section demonstrates an example of this process using a Domineering game tree.

Solving a Domineering Game

A Domineering game is considered solved if one can find which player has a winning strategy for each turn order. We will show an example of how to solve a Domineering game by constructing a game tree for the Domineering game in Diagram 1-20.

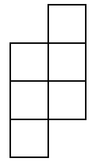


Diagram 1-21

There are a few things to be aware about constructing a combinatorial game tree. The first is that symmetric moves are only included once in a game tree. Diagrams 1-22a and 1-22b show an example of symmetric moves.

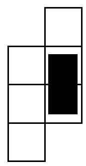


Diagram 1-22a

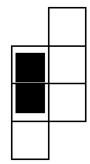


Diagram 1-22b

After L places his vertical piece in both Diagrams 1-22a and 1-22b, one can see that the boards' empty spaces are equal when you account for symmetry.

The second is that any covered squares or isolated squares (with no other side-adjacent squares) that neither player can play are ignored. Thus, the result of Diagrams 1-22a and 1-22b can be written as follows:

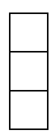


Diagram 1-23

The exception to this rule is when the game ends. If the result of a position is just an isolated square (or several separate isolated squares), then one isolated square can be listed as that position's child. Let's construct the entire game tree for the game in Diagram 1-21.

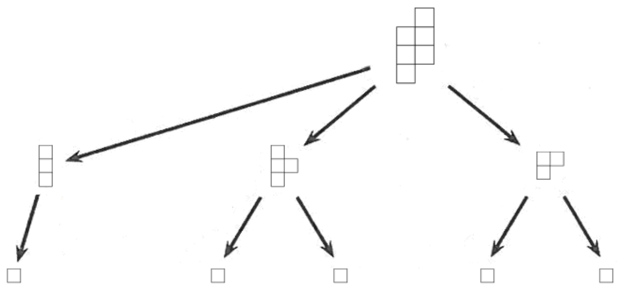


Diagram 1-24

Once the game tree is constructed, the next goal to solving a Domineering game is to find the outcome class of the game tree's root position. The first step of this process is to calculate the outcome class of the tree's leaves. In Diagram 1-24, all the leaves are single isolated games (or 0 games), and those games are $\mathcal{P}$ because the next player is unable to place a piece in his turn, resulting in a loss for that player. So let's replace those leaves with $\mathcal{P}$.

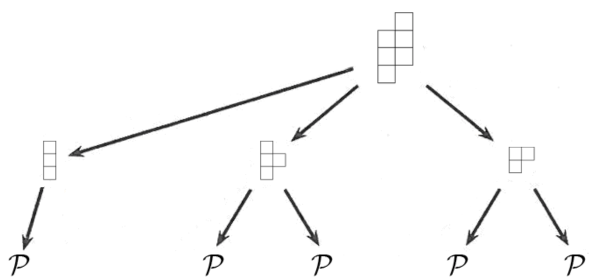


Diagram 1-25

The next step is to calculate the outcome classes of the positions above the leaves. The following chart in Diagram 1-26, defined in *Lessons in Play*, explains how to calculate the outcome class of a position based on the outcome classes of its children [ANW_07].

How to Determine an Outcome Class for a Combinatorial Game Position		
	some right option $\in \mathcal{R} \cup \mathcal{P}$	all right options $\in \mathcal{L} \cup \mathcal{N}$
some left option $\in \mathcal{L} \cup \mathcal{P}$	$\mathcal{N}$	$\mathcal{L}$
all left options $\in \mathcal{R} \cup \mathcal{N}$	$\mathcal{R}$	$\mathcal{P}$

Diagram 1-26

For example, the chart says that if at least one left option and at least one right option of a position are $\mathcal{P}$, then the position itself is $\mathcal{N}$, so we'll replace the positions above the two $\mathcal{P}$ options in Diagram 1-25 with $\mathcal{N}$. Diagram 1-25 also has a three-space column game with no right children, which means that the game's set of right options is the empty set. The outcome class for that position is $\mathcal{L}$ since it has one left option in $\mathcal{L}$ and the "all right options $\in \mathcal{L} \cup \mathcal{N}$" is true because there are no right options.

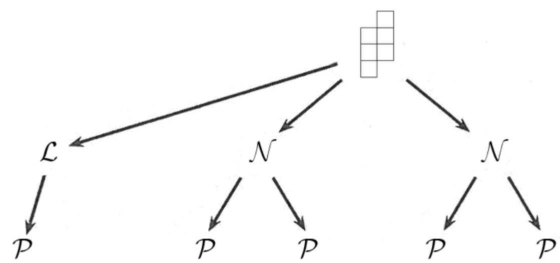


Diagram 1-27

This process is repeated as you move up the game tree until you ultimately calculate the outcome class of the game tree's root position. Let's analyze the root position in Diagram 1-25 using the chart from Diagram 1-26.

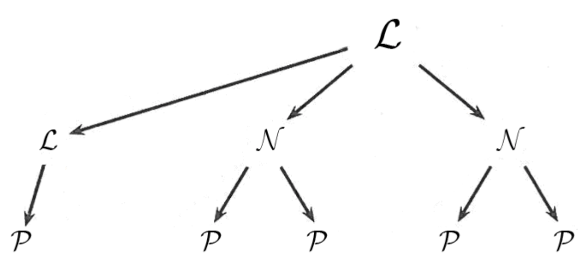


Diagram 1-28

Diagram 1-28 shows that after calculating every position's outcome class, the root position's outcome class is $\mathcal{L}$. This means that left player can force a win regardless of whoever goes first. The reader can see in Diagram 1-26 which moves would guarantee a victory for left player for both turn orders (those moves are left player's winning strategy). Thus, the Domineering game in Diagram 1-21 is solved.

Solved Domineering Games

The larger a Domineering board is, the more branches its game tree will have, making it harder to determine the game's outcome class by hand. Artificial intelligence researchers have looked into solving the larger boards by constructing search engines to calculate their outcome class. These search engines use a combination of alpha-beta pruning and a move ordering heuristic to remove as many unnecessary branches as possible from the engine's traversal and analysis of the game tree [BUH_00, Bul_02].

Brueker, Uiterwijk and van den Herik created the program DOMI to solve m x n boards where $2 \le m \le 8$ and $m \le n \le 9$ [BUH_00]. Bullock later improved on their research by developing a search application called Obsequi that could not only solve those games faster, but also solve a 10 x 10 board, which is currently the largest solved Domineering game [Bul_02].

Two-player combinatorial games and 2D Domineering have been researched extensively by mathematicians and computer scientists. The three-player variants of those games, on the other hand, have not been explored as much. In the next section, we will introduce three-player combinatorial games, specifically 3D Domineering, and the steps needed to solve a 3D Domineering game.

1.2 – Three-Player Combinatorial Games

We defined combinatorial games in the previous section as two-player games, but there exist variants of those games that can be played with three players. These three-player games can still be considered combinatorial games as long as they have the following properties: alternating turns, no ties, loss for player with no more legal moves, perfect information, no elements of chance, and no cycles.

The addition of the third player makes analyzing combinatorial games more complex and complicated. For example, to construct a game tree for a three-player combinatorial game, the children of each position must be divided into three distinct sets, one for the left player, one for the middle player, and one for the right player (as opposed to two distinct sets for two-player trees). Options for the left, middle, and right player are drawn as a position's left, middle, and right children respectively. The following diagram is an example of a game tree for a three-player combinatorial game, with the first, middle, and right options marked as 'L', 'M', and 'R' respectively.

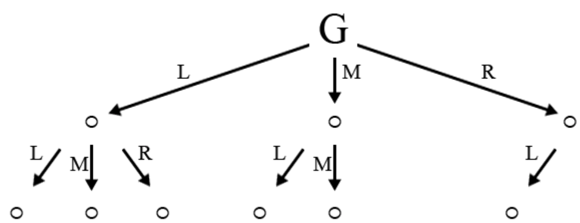


Diagram 1-29
An example of a three-player combinatorial game tree

Outcome Classes for Three-Player Games

Another aspect of combinatorial games that changes with the addition of a third player is their outcome classes. The outcomes classes that were derived from the Fundamental Theorem for two-player combinatorial games (where exactly one player has a winning strategy for each turn order) do not apply to three-player combinatorial games. This is because it is possible in a three-player game that no player can have a winning strategy in a turn order. Philip Straffin explored this possibility through a concept introduced in his paper called "decision by player" [Str_85]. A "decision by player" means that one player who is unable to win a three-player impartial game can affect which of the other two players can win with his next move. Thus, it is possible that one player cannot force a win no matter what he does if one of his opponents plays a move that causes the other opponent to win.

James Propp defined the scenario where no player can force a win as "queer" or $\mathcal{Q}$ for three-player impartial games [Pro_00]. Alessandro Cincotti also used the term to describe the same scenario for three-player partizan games [Cin_08]. We suggest that we change this term to "Qual," which still maintains the $\mathcal{Q}$ moniker that fits with the other outcome classes defined for three-player games (such as $\mathcal{N}$, $\mathcal{O}$, and $\mathcal{P}$, which we'll introduce a bit later). Qual is German for "torment" and is used in the German saying, "Wer die Wahl hat, hat die Qual," which translates to, "Whoever has the choice, has the torment." This saying seems appropriate for games where no one can force a win as whoever has the first choice in a $\mathcal{Q}$ game also has the torment of not having a winning strategy. The following is an example of a $\mathcal{Q}$ game using Nim.

❍
❍ ❍

Diagram 1-30
A Nim game in outcome class $\mathcal{Q}$

The Nim game in Diagram 1-30 is $\mathcal{Q}$ because none of the three players can force a win. The first player cannot win because he cannot remove both heaps in his first turn, and what's left will be taken by the other two players before the first player's next turn. The second player cannot win if the first player removes one counter from the two-counter heap, leaving the second player with two heaps. The second player can only remove one heap in his next turn, leaving the other heap for the third player to remove and win. The third player cannot win if the first player removes one heap stack from the game and the second player removes the other heap stack.

James Propp also defined the following outcome classes for three-player impartial games: $\mathcal{N}$ means that the first player (or next player) can force a win. $\mathcal{O}$ means that the second player (or other player) can force a win. $\mathcal{P}$ means that the third player (or previous player) can force a win. The following Nim games show examples of these outcome classes.

❍

Diagram 1-31
A Nim game in outcome class $\mathcal{N}$

The Nim game in Diagram 1-31 is $\mathcal{N}$ because the first player can always win simply by removing the sole counter.

❍ ❍

Diagram 1-32
A Nim game in outcome class $\mathcal{O}$

The Nim game in Diagram 1-32 is $\mathcal{O}$ because the second player can always win by removing the remaining heap after the first player removes one of the heaps.

❍ ❍ ❍

Diagram 1-33
A Nim game in outcome class $\mathcal{P}$

The Nim game in Diagram 1-33 is $\mathcal{P}$ because the third player can always win by removing the last heap after the first and second players remove a heap in their respective turns.

Propp established rules for classifying a three-player impartial game using a game tree.

- A game is $\mathcal{N}$ if at least one of its options is $\mathcal{P}$.
- A game is $\mathcal{O}$ if all its options are $\mathcal{N}$ and has at least one option.
- A game is $\mathcal{P}$ if all its moves are $\mathcal{O}$ games.
- A game is $\mathcal{Q}$ if none of the above applies.

These outcome classes can also define three-player partizan games, but more outcome classes are needed because left, middle, and right players have different moves in partizan games. Alessandro Cincotti defined 27 outcome classes for three-player partizan games when those games are defined as numbers using inequalities he introduced in his paper [Cin_05]. Matt Klassen, however, suggests that there are more basic outcome classes for three-player partizan games, 4,096 to be exact [Kla_10], which we will explain below.

There are four possible results for each of the six turn orders after playing through a three-player game: either left player (L) wins, right player (R) wins, middle player (M) wins, or no player (N) can force a win, a total of 4^6 = 4,096 outcome classes. In the chart below, we will show eight of the 4,096 possible outcome classes for a three-player partizan combinatorial game.

Sample of Outcome Classes for Three-Player Partizan Combinatorial Games								
Turn Order	Outcome Classes							
	$\mathcal{N}$	$\mathcal{O}$	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{M}$	$\mathcal{Q}$	?
LMR	L	M	R	L	R	M	N	M
LRM	L	R	M	L	R	M	N	N
MLR	M	L	R	L	R	M	N	R
MRL	M	R	L	L	R	M	N	M
RLM	R	L	M	L	R	M	N	L
RML	R	M	L	L	R	M	N	N

Diagram 1-34

In the Diagram 1-34 chart, we listed 'L,' 'M,' 'R,' or 'N' for each outcome class' turn order. Those represent which of the three players has a winning strategy for that turn order (or in N's case, none of the players have a winning strategy). Note that we introduced the $\mathcal{M}$ outcome class, where middle player can force a win in each turn order.

The reader may notice that we listed the last outcome class as '?' on the table. This is because, unlike the other seven outcome classes in Diagram 1-34, that outcome class does not have an obvious label. The same can be said for most of the other 4,088 outcome classes not defined in this table. It is important to note, however, that all 4,096 outcome classes are non-empty, meaning that there is always a specific game for each outcome class. To assist in demonstrating this, we will introduce L_0, M_0, R_0, and N_0.

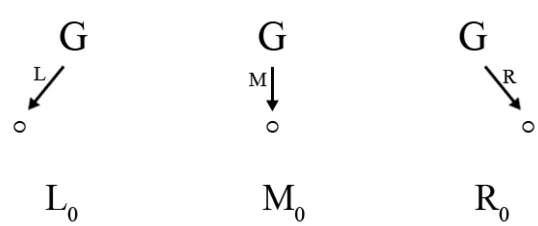


Diagram 1-35

L_0, M_0, and R_0 are specific three-player games where L, M, and R have a winning strategy respectively. N_0 is defined as the following game.

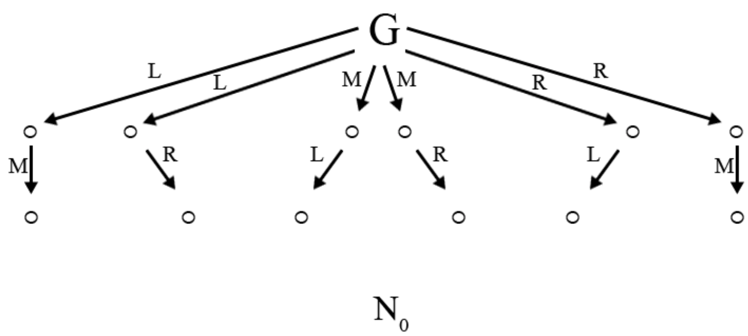


Diagram 1-36

The N_0 tree indicates that no one can force a win, regardless of whether the player goes first, second, or third. In other words, $L_0 \in \mathcal{L}$, $M_0 \in \mathcal{M}$, $R_0 \in \mathcal{R}$, and $N_0 \in \mathcal{Q}$.

These four games can be used to define any of the 4,096 outcome classes we introduced earlier and show that none of them are empty games. We will demonstrate an example of this by using the four games to define the '?' outcome class listed in the Diagram 1-34 table.

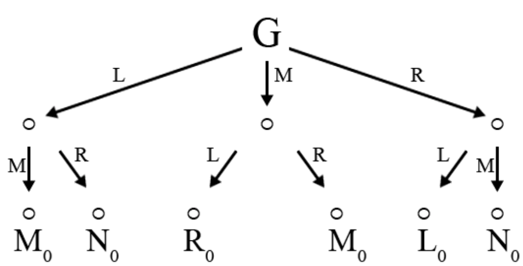


Diagram 1-37

If you replace L_0, M_0, R_0, and N_0 in Diagram 1-37 with the game trees we defined in Diagrams 1-34 and 1-35, you have created a game tree that defines who wins at each turn order of the '?' outcome class. The reader should note that the order of L_0, M_0, R_0, and N_0 in the second level of Diagram 1-37's game tree is the same as the order of results listed for each turn order of the '?' outcome class in Diagram 1-34. This process can be done for all 4,096 outcome classes, meaning that all of them are non-empty.

The large number of outcome classes makes it more difficult to solve a three-player game via its game tree. This is because there are too many outcome classes to create a chart similar to the one in Diagram 1-26 that calculates the outcome class of a tree's position based on its children. However, it is still possible to solve a three-player combinatorial game by showing for each turn order a player's winning strategy or that no player can force a win.

In the following pages, we'll explain the three-player combinatorial game, 3D Domineering, and how to solve that particular game.

What is 3D Domineering?

3D Domineering is a three-player variant of the Domineering game. It is played on a rectangular solid board similar to a Rubik's cube whose edges are parallel to the x, y, and z axes. The three players have pieces that occupy two adjacent cubes of the board and take turns in a cyclical order placing them in the 3D board. One player is limited to placing pieces along the x-axis of the board, another player is limited to placing pieces along the y-axis of the board, and the remaining player is limited to placing pieces along the z-axis of the board. If a player is unable to place any more pieces in the board at his turn, then that player is eliminated from the game. The remaining two players will continue to play until one is unable place any more pieces in the board, resulting in a loss for that player.

Example of a 3x3x3 Domineering Game

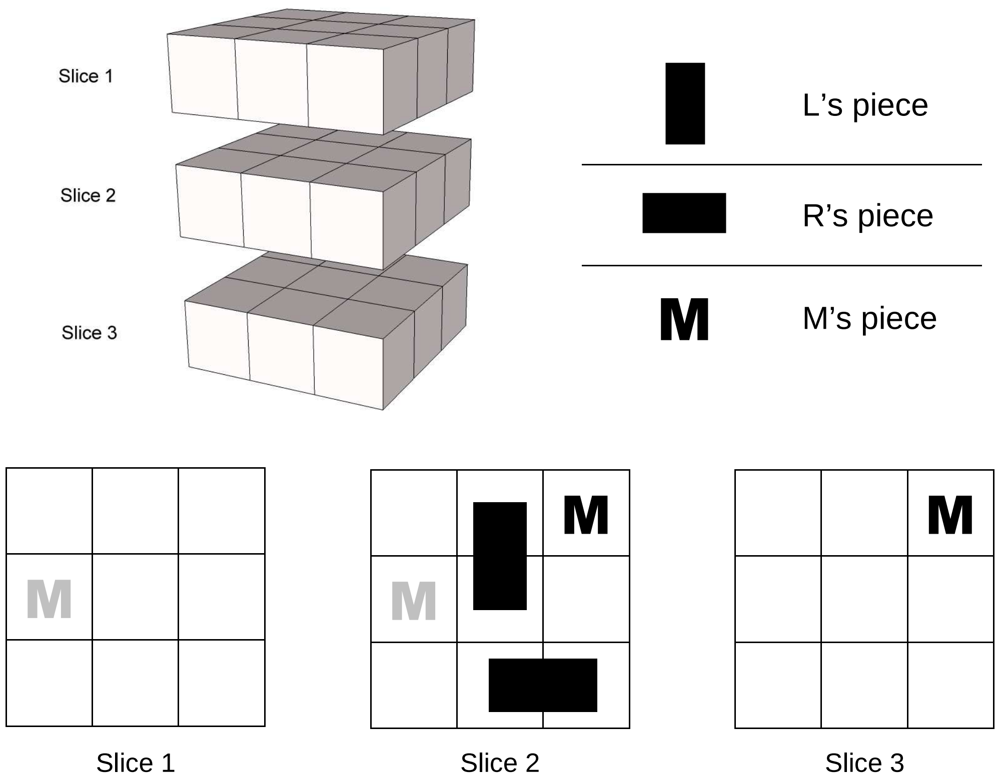


Diagram 1-38

Diagram 1-38 shows an example of a 3x3x3 Domineering game. We marked the three players as L, R, and M. Let’s suppose that the turn order for this game is M-L-R. M plays a piece that traverses Slices 1 & 2 on the board. Then L and R play their pieces on Slice 2. M follows up by playing a piece that traverses Slices 2 & 3. All three players continue to take turns until two of the players are no longer able to place any more pieces.

Solving a 3D Domineering Game

A 3D Domineering game is considered solved when, for each turn order, one can either identify the winning strategy for a player or prove that no player has a winning strategy.

Alessandro Cincotti was able to solve many A x B x C 3D Domineering games where $A + B + C < 10$ and $A, B, C \geq 2$ using an exhaustive search algorithm [Cin_08, Cin_09]. His results are shown in the following chart (the chart presumes L, R, and M play on A, B, and C respectively).

Outcome Classes for Three-Player Domineering			
	L Playing First	R Playing First	M Playing First
2x2x2	$\mathcal{M}$	$\mathcal{L}$	$\mathcal{R}$
3x2x2	$\mathcal{Q}$	$\mathcal{L}$	$\mathcal{Q}$
2x3x2	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{R}$
2x2x3	$\mathcal{M}$	$\mathcal{Q}$	$\mathcal{Q}$
3x3x2	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{M}$
2x3x3	$\mathcal{L}$	$\mathcal{Q}$	$\mathcal{Q}$
3x2x3	$\mathcal{Q}$	$\mathcal{R}$	$\mathcal{Q}$
3x3x3	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$
4x2x2	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$
2x4x2	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$
2x2x4	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$
4x3x2	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$
2x4x3	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$
3x2x4	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$
4x2x3	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$
3x4x2	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$
2x3x4	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$
5x2x2	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$
2x5x2	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$
2x2x5	$\mathcal{Q}$	$\mathcal{Q}$	$\mathcal{Q}$

Diagram 1-39

As of this writing, no analysis of 3D Domineering games where $A + B + C \geq 10$ and A, B, $C \geq 2$ has been published yet. This is probably because the number of branches in a 3D Domineering game tree becomes exponentially larger and more computationally expensive to traverse as the 3D game board's size increases. However, we have found strategies that can significantly reduce the number of branches to traverse in a 3D Domineering game tree.

Specifically, we found strategies for two players that can prevent the third player from winning, regardless of the turn order and the third player's actions. Selecting specific moves for the two players, instead of considering all their possible moves, allows us to prune branches from the 3D Domineering game tree. This makes it faster to traverse the game tree and find the outcome class for 3D Domineering games, especially for those whose sum of its dimensions is greater than or equal to ten. In the next chapter, we will demonstrate how these strategies prove that a 3x3x3 Domineering game is a $\mathcal{Q}$ game (no player can force a win in all turn orders). In subsequent chapters, we will expand on those strategies to show that a 4x4x3 game, a game that has not been solved before, is also a $\mathcal{Q}$ game.

2.0 – Strategies to Solve a 3x3x3 Domineering Game

In this chapter, we will present the strategies that will prove that a 3x3x3 Domineering game is a $\mathcal{Q}$ game, where no one can force a win regardless of turn order. Alessandro Cincotti had already solved the 3x3x3 Domineering game by using an exhaustive search algorithm [Cin_08], but we will still present our strategies because they will serve as the foundation for our proof to solve a 4x4x3 game.

In this chapter, we are going to prove the following:

Proposition 2-1: *In a 3x3x3 game of Domineering, it is impossible for the middle player (M) to force a win if both left player (L) and right player (R) collude to stop M.*

Since a 3x3x3 game is symmetrical on all sides, if we prove that M cannot force a win if L and R collude to stop M, then it also means that L cannot force a win if R and M collude to stop L, and R cannot force a win if L and M collude to stop R. Thus, proving Proposition 2-1 will also prove that no player can force a win in a 3x3x3 Domineering game.

2.1 – Strategy to Block M in a 3x3x3 Domineering Game

There exists a strategy that L & R can follow that will prevent M from winning regardless of whether M plays first, second, or third. To be clear, a win for M in a three-player Domineering game means that M still has at least one legal move left after both L and R have no more legal moves. There are two stages to this blocking strategy, the first which we will demonstrate using the following examples of a 3x3x3 game:

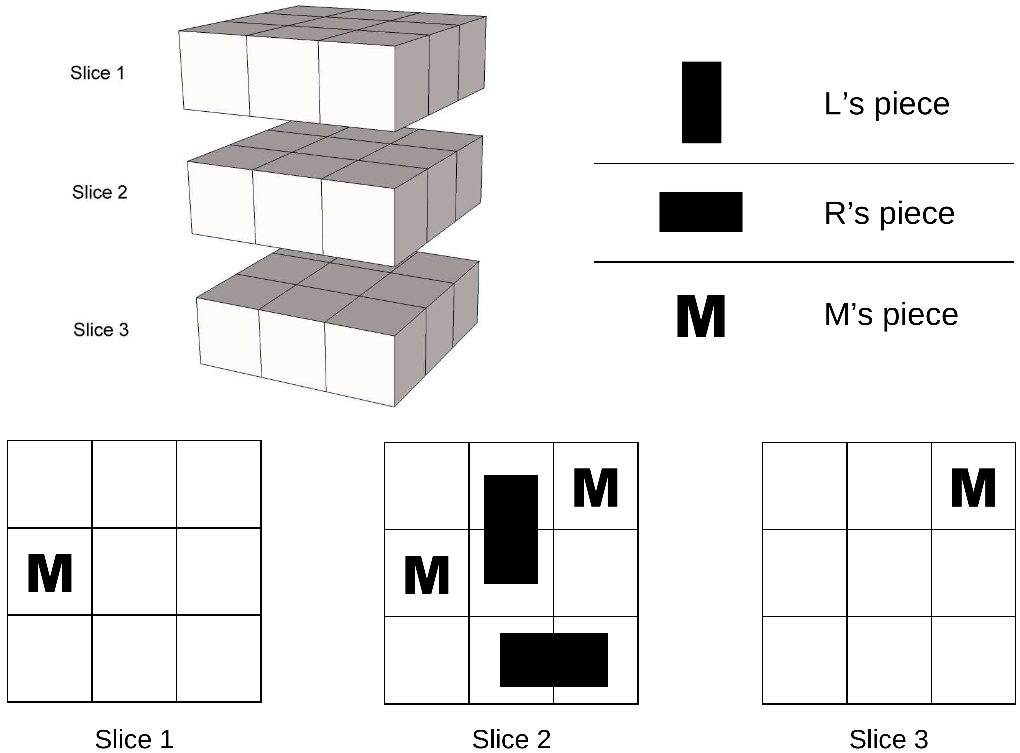


Diagram 2-1
An example of a 3x3x3 Domineering game

Notice that when M places a piece, it traverses either Slices 1 and 2 or Slices 2 and 3.

Observation: *If all the spaces in Slice 2 are occupied, then M can no longer make any more moves.*

Therefore, the first part of the blocking strategy that L and R should adopt to prevent M from winning is as follows:

Strategy: *L & R should place as many pieces as possible on Slice 2 first before placing any pieces on Slices 1 or 3.*

Since L and R are going to be working together to stop M, we should consider this scenario as a special two-player game between M and the L & R alliance. This means that if M is unable to play any legal moves before both L and R are eliminated, then the alliance wins. Furthermore, the alliance has an advantage over M where the alliance can continue to play even if one of its members is unable to place any more pieces.

For this special two-player game, M's move is considered a half-turn, and the other half-turn is considered complete when the alliance finishes their turn. This holds true whether L & R are both playing, or if only one player in the alliance is left playing against M. Thus, if both L & R are playing, their moves are considered ¼ turns. However, if only one member in the alliance is still playing, then those moves are considered half-turns.

We will demonstrate by example the alliance's strategy to cover as much of Slice 2 as possible using the following L-R-M game.

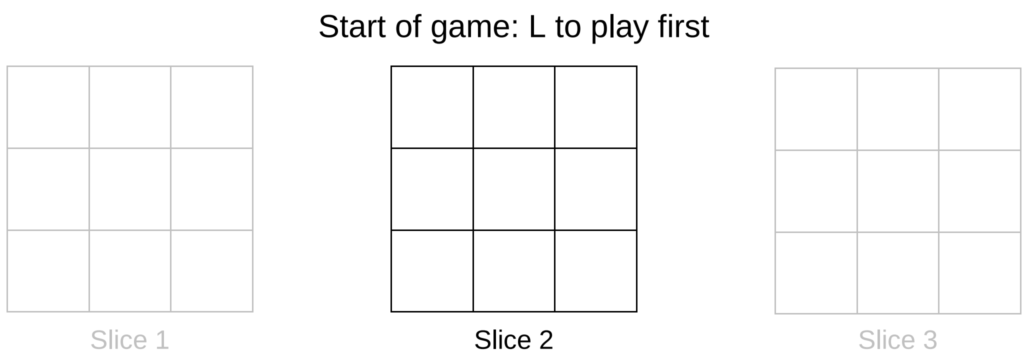


Diagram 2-2
The start of a 3x3x3 L-R-M Domineering Game

At the start of the game, Slice 2 has nine available squares, as shown in Diagram 2-2.

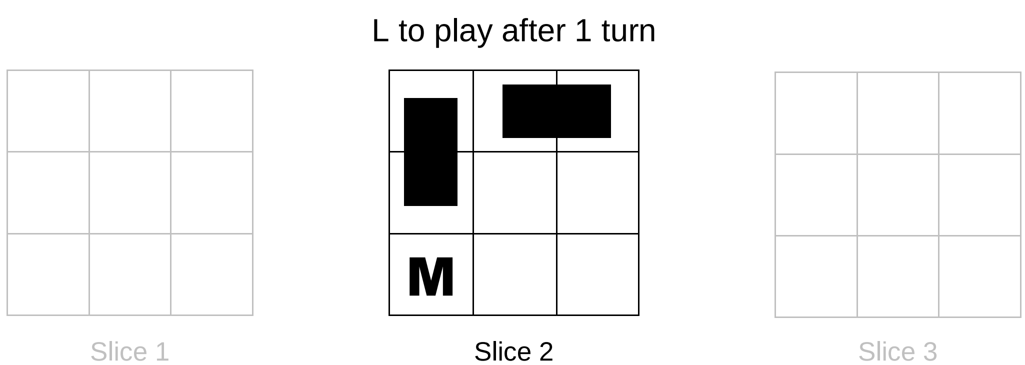


Diagram 2-3
A 3x3x3 Domineering game after one turn.
The other M piece is on Slice 1 or Slice 3, but for this example, its location is not important.

On the first turn (Turn 1), L, R, and M will each place a piece on Slice 2. At the end of Turn 1, there are four spaces left in Slice 2 as shown in Diagram 2-3.

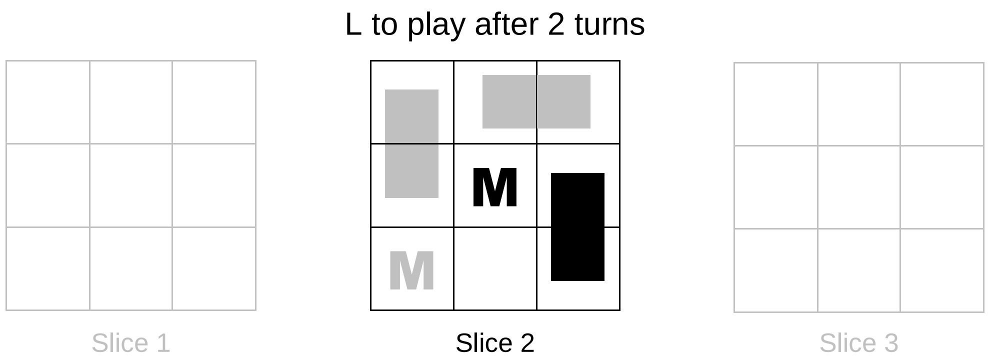


Diagram 2-4
A 3x3x3 Domineering game after two turns.
Two one-space M pieces and an R piece are on Slice 1 and/or Slice 3

After the second turn (Turn 2), either L or R (in this example R) will be unable to place a piece on Slice 2, forcing that player to play on Slice 1 or 3. The other two players will place their pieces on Slice 2, leaving just one space left in Slice 2. At the start of the third turn (Turn 3), M has only one more move available.

At this point, the alliance can no longer move on Slice 2, but it can still block M with the second part of its blocking strategy:

Strategy: *When there are no more available moves in Slice 2 for the alliance, L and R can cooperatively block M's remaining moves by placing pieces above and below M's available spaces on Slice 2.*

We'll use one possible game from Diagram 2-4 to demonstrate this strategy.

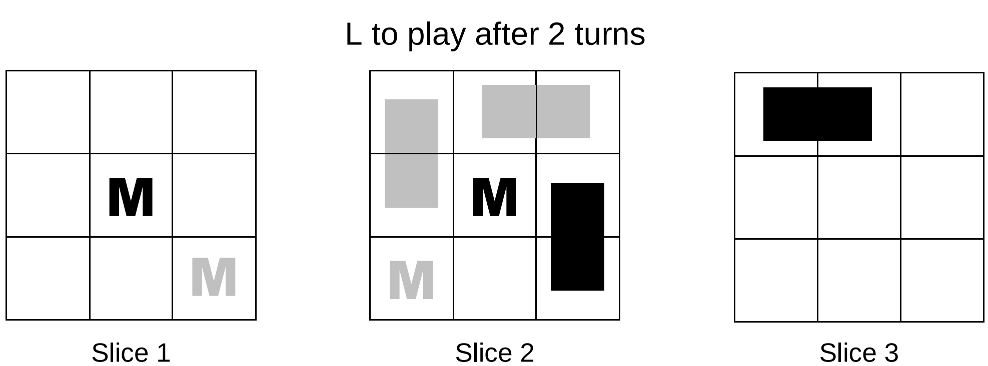


Diagram 2-5
One possible game from Diagram 2-4.

In Diagram 2-5, M has one space left in Slice 2 and the alliance have an opportunity to block that last space with their next move.

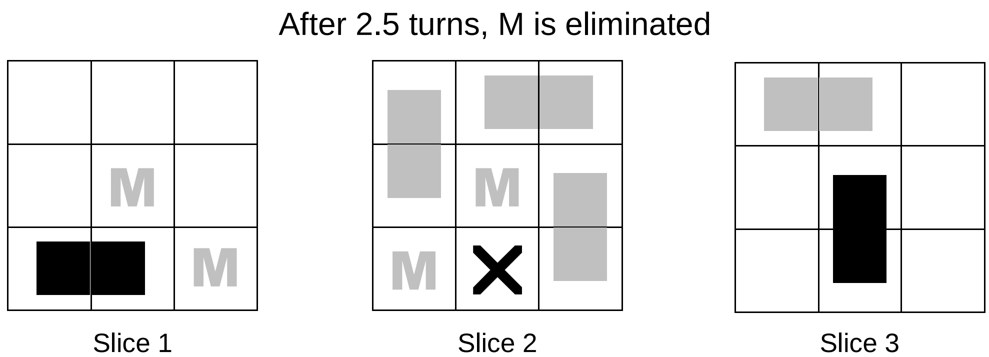


Diagram 2-6
The alliance cooperatively blocks M's last remaining move (as indicated by the 'X').

Diagram 2-6 shows that the alliance's next moves block M from playing on the last available space on Slice 2 (indicated by the 'X'). M has no more available moves, which results in a loss for M.

We have shown in the previous example how the alliance's blocking strategy can prevent M from winning, but our previous example only covers one possible 3x3x3 Domineering game. To prove Proposition 2.1, we will need to show that the alliance's two-part blocking strategy prevents M from winning in all possible 3x3x3 games.

In the following sections, we'll show specific moves that the alliance can play in a 3x3x3 Domineering game to win against M. For our examples, we'll show all possible moves for M, unless they can be ignored because of symmetry or because they would lead to an immediate loss for M.

We begin our analysis with 3x3x3 games where M goes last.

2.2 – Analysis of L-R-M and R-L-M Games

In this section, we continue the proof of Proposition 2-1 by showing that the alliance can prevent M from winning in 3x3x3 Domineering games where M goes last. We will start by proving the following for 3x3x3 games:

Proposition 2-2: *In L-R-M and R-L-M turn order games, the alliance can limit M to at most one space in Slice 2 after two complete turns.*

Proof:

Let's examine how this is possible.

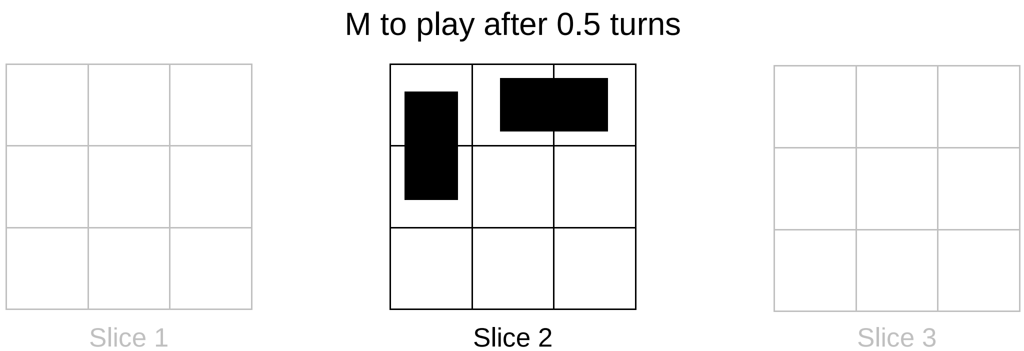


Diagram 2-7
The alliance's first moves on Slice 2.

For L-R-M and R-L-M games, Diagram 2-7 demonstrates the first possible moves the alliance can make. We will show in subsequent examples that these moves are a good first move for the alliance because M will not be able to force a win.

There are five possible ways that M can react to the alliance's first moves in Diagram 2-7, as demonstrated in the following diagram.

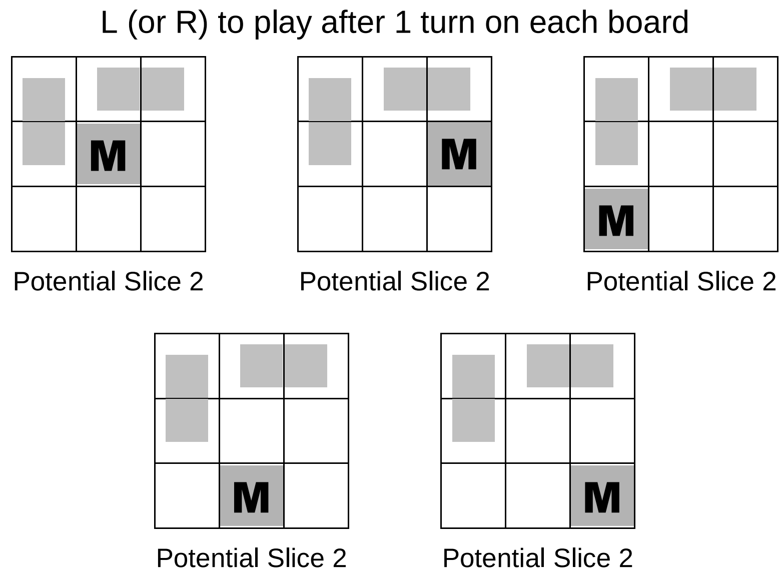


Diagram 2-8
Each shaded square represents a possible move M can make after Diagram 2-7.

After M makes his first turn, the reader can see that in each possible game of Diagram 2-8, there are four available spaces left in Slice 2. M playing on the center square in his first turn (the first game in Diagram 2-8) is a really bad move for M because L and R are able to cover the remaining spaces of Slice 2 in their next turns, eliminating M after 1.5 turns.

In the other four games of Diagram 2-8, one can see that only one of the alliance members can move in the remaining spaces of Slice 2 in his next turn. The other alliance member is forced to play on either Slice 1 or Slice 3. After either L or R moves on Slice 2 in those games, there are two available spaces left for M. When M moves on one of those spaces, there will be one space left on Slice 2 after two complete turns. □

This concludes the proof of Proposition 2.2.

Although all the alliance needs to do after two turns is cooperatively block M's remaining move to eliminate M, we can actually prove the following:

Proposition 2-3: *After the alliance limits M to one space in Slice 2 after two complete turns, the alliance has at least one more move after M plays his final move even if L and R do not cooperatively block M's last move.*

Proof:

The following chart shows the number of available spaces in each slice at every turn of a specific L-R-M and R-L-M game.

Number of free spaces in each turn of a specific L-R-M & R-L-M game						
Turn Number	L-R-M Game: Who Plays Next	R-L-M Game: Who Plays Next	# of Free Spaces in Slice A	# of Free Spaces in Slice 2	# of Free Spaces in Slice B	Total # of Free Spaces
0	L to play next	R to play next	9	9	9	27
0.25	R to play next	L to play next	9	7	9	25
0.5	M to play next	M to play next	9	5	9	23
1	L to play next	R to play next	8	4	9	21
1.25	R to play next	L to play next	8	2	9	19
1.5	M to play next	M to play next	8	2	7	17
2	L to play next	R to play next	7	1	7	15
2.25	R to play next	L to play next	5	1	7	13
2.5	M to play next	M to play next	5	1	5	11
3	L to play next	R to play next	4	0	5	9

Diagram 2-9

There are a couple of things to note about the chart. First, the first part of the alliance's blocking strategy (L and R covering as much of Slice 2 as possible) is reflected in the moves made in the chart. Second, for the columns "# of Free Spaces in Slice A" and "# of Free Spaces in Slice B", they can either be A = 1 and B = 3 or A = 3 and B = 1. Third, the "Total # of Free Spaces" column can refer to all possible L-R-M and R-L-M games, even if the chart refers to one specific game for each turn order. For example, let's say that after 1.5 turns, M decides to play on Slices 2 and B instead of Slices 2 and A.

Turn Number	L-R-M Game: Who Plays Next	R-L-M Game: Who Plays Next	# of Free Spaces in Slice A	# of Free Spaces in Slice 2	# of Free Spaces in Slice B	Total # of Free Spaces
2	L to play next	R to play next	8	1	6	15
2.25	R to play next	L to play next	6	1	6	13
2.5	M to play next	M to play next	6	1	4	11
3	L to play next	R to play next	6	0	3	9

Diagram 2-10

Although the number of free spaces in Slices A and B are different in Diagram 2-10 than what's listed in Diagram 2-9, notice that the Total # of Free spaces are the same in both charts at each turn order. This is because every piece each player places takes up two spaces on the board. Regardless of how L, R, and M move in L-R-M and R-L-M games, the total number of free spaces at each turn order will always be the same. Thus, after the end of the second turn in L-R-M and R-L-M games, there will always be nine available spaces left (assuming that L & R do not cooperatively block M's remaining move).

The best way to prove that M loses in a three-player game is to show that either L or R has at least one more move after M plays his final piece. Even if L and R do not block M's last move during their second turn, there are still nine available spaces between Slices 1 and 3 after M plays his last move. With nine available spaces, it is guaranteed that either L or R will be able to move at least once after M's final turn, which is a loss for M. □

This concludes the proof of Proposition 2.3.

The proof of Proposition 2.3 shows that the alliance can prevent M from winning 3x3x3 games where M goes last. The next section discusses 3x3x3 games where M goes second.

2.3 – Analysis of L-M-R and R-M-L Games

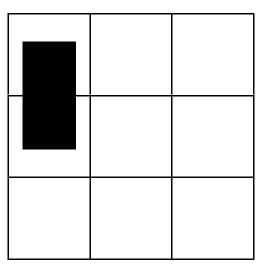
In this section, we continue the proof of Proposition 2-1 by showing that the alliance can prevent M from winning in 3x3x3 Domineering games where M goes second. We will start by proving the following for 3x3x3 games:

Proposition 2-4: *In L-M-R and R-M-L turn order games, the alliance can limit M to at most two spaces in Slice 2 after 1.75 turns.*

Proof:

Diagrams 2-11a and 2-11b show examples of L and R's first moves in an L-M-R and an R-M-L game respectively.

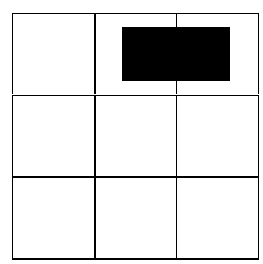
M to play after 0.25 turns



Slice 2 (L-M-R game)

Diagram 2-11a
L's first move

M to play after 0.25 turns



Slice 2 (R-M-L game)

Diagram 2-11b
R's first move

We will show in subsequent examples that these moves are a good first move for the alliance because M will not be able to force a win. The following examples in this section show an L-M-R game, but they can also correlate to an R-M-L game because of rotational symmetry (the reader can confirm this by looking at Diagrams 2-11a and 2-11b).

There are seven possible ways that M can react to L's (or R's) first move in an L-M-R (or R-M-L) game, which are shaded in the seven games of the following diagram. We also demonstrate how the alliance can react to each of M's seven possible first moves.

M to play after 1.25 turns on each board (all represent Slice 2)

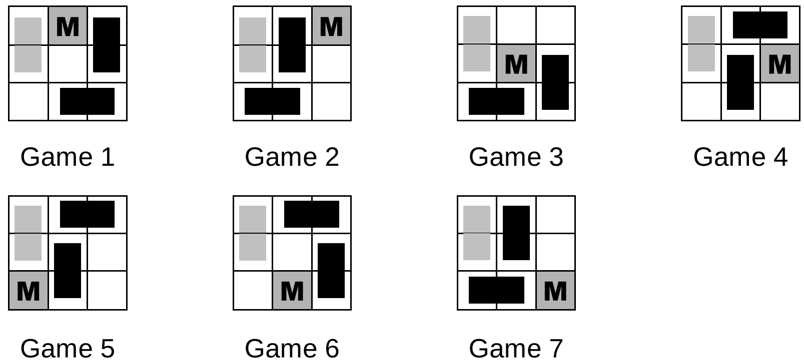


Diagram 2-12
M's seven possible first moves and the alliance's reaction to those moves.

Note that in each of the seven games in Diagram 2-12, there are two remaining spaces left in Slice 2. After M moves on one of those spaces in his next turn, there will be just one space left after 1.75 turns. □

This concludes the proof of Proposition 2-4.

Although all the alliance needs to do after 1.75 turns is cooperatively block M's remaining move to eliminate M, we can actually prove the following:

Proposition 2-5: *After the alliance limits M to one space in Slice 2 after 1.75 turns, the alliance has at least one more move after M plays his final move even if L and R do not cooperatively block M's last move.*

Proof:

To prove this proposition, we will use a chart similar to the one in Diagram 2-9 to show the number of available spaces left in each slice after each turn in a specific L-M-R and R-M-L game.

Number of free spaces in each turn of a specific L-M-R & R-M-L game						
Turn Number	L-M-R Game: Who Plays Next	R-M-L Game: Who Plays Next	# of Free Spaces in Slice A	# of Free Spaces in Slice 2	# of Free Spaces in Slice B	Total # of Free Spaces
0	L to play next	R to play next	9	9	9	27
0.25	M to play next	M to play next	9	7	9	25
0.75	R to play next	L to play next	8	6	9	23
1	L to play next	R to play next	8	4	9	21
1.25	M to play next	M to play next	8	2	9	19
1.75	R to play next	L to play next	7	1	9	17
2	L to play next	R to play next	7	1	7	15
2.25	M to play next	M to play next	5	1	7	13
2.75	R to play next	L to play next	4	0	7	11

Diagram 2-13

There are a couple of things to note about the chart. First, the first part of the alliance's blocking strategy (L and R covering as much of Slice 2 as possible) is reflected in the moves made in the chart. Second, for the columns "# of Free Spaces in Slice A" and "# of Free Spaces in Slice B", they can either be A = 1 and B = 3 or A = 3 and B = 1. Third, the "Total # of Free Spaces" column can refer to all possible L-M-R and R-M-L games even if the chart refers to one specific game for each turn order. Recall that we demonstrated in the Diagram 2-9 and 2-10 charts that, regardless of where L, R, and M play, each player's piece takes up two spaces.

To reiterate what was stated in the last section, the best way to prove that M loses in a three-player game is to show that either L or R has at least one more move after M plays his final piece. Even if L and R do not block M's last move after 1.75 turns, there are still eleven available spaces between Slices 1 and 3 after M plays his last move. With eleven available spaces, it is guaranteed that either L or R will be able to move at least once after M's final turn, which is a loss for M. □

This concludes the proof of Proposition 2.5.

The proof of Proposition 2.5 shows that the alliance can prevent M from winning 3x3x3 games where M goes second. The next section discusses 3x3x3 games where M goes first.

2.4 – Analysis of M-L-R and M-R-L Games

We have proven in the last two sections that the alliance can prevent M from winning 3x3x3 Domineering games whenever M goes second or third. If we prove that the same applies to games where M goes first, then we have demonstrated that the alliance can prevent M from winning all possible 3x3x3 games. In this section, we conclude the proof of Proposition 2-1 by proving the following for 3x3x3 games:

Proposition 2-6: *In M-L-R and M-R-L turn-order games, the alliance can limit M to a maximum of three spaces in Slice 2 after two turns.*

Proposition 2-7: *In M-L-R and M-R-L turn-order games, the alliance has at least one more move after M plays his final piece.*

There are nine starting moves for M in 3x3x3 games where M goes first. However, we can break those down to three types of moves, as demonstrated in Diagram 2-14.

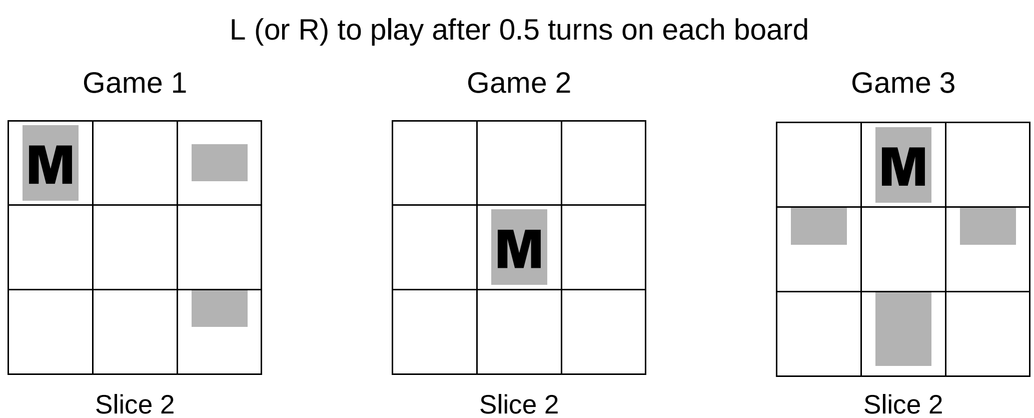


Diagram 2-14
The three types of moves M can play when he goes first.

The first type of move M can make is playing on a corner space. M playing on any of the shaded spaces in Game 1 of Diagram 2-14 represents the same move because of rotational symmetry. The second type of move M can make is playing on the center space, as shown in Game 2 of Diagram 2-14. The third type of move M can make is playing on a space that is neither corner nor center. M playing on any of the shaded squares in Game 3 of Diagram 2-14 represents the same move because of rotational symmetry. Therefore, we only need to show these three games in our subsequent examples.

We will continue the proof of Proposition 2-6 with demonstrating the first two types of moves M can make.

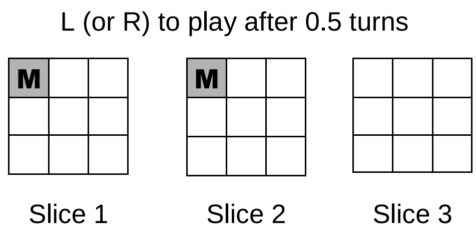


Diagram 2-15a
M plays on the upper-left corner.

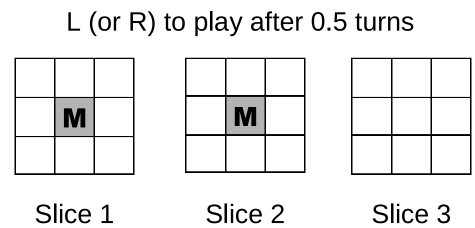


Diagram 2-15b
M plays on the center square.

In Diagram 2-15a, M plays on a corner space. In Diagram 2-15b, M plays on the center space. In both games, M's piece traverses Slices 1 and 2, but M could also place a piece that traverses Slices 2 and 3. However, we do not need to show the games where M's piece traverses Slices 2 and 3 because they are symmetrical to the games where M's piece traverses Slices 1 and 2.

The following shows how the alliance can respond to M's corner and center moves.

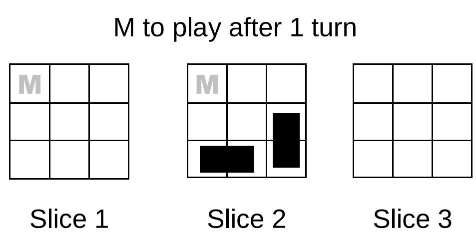


Diagram 2-16a
The alliance responds to M's corner move.

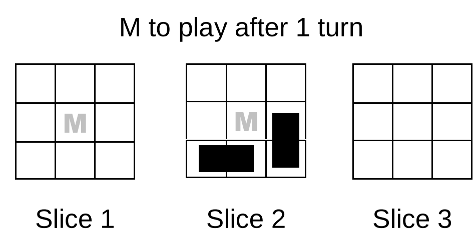
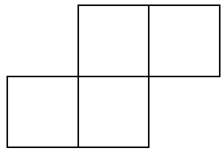


Diagram 2-16b
The alliance responds to M's center move.

We will show in subsequent examples that these moves are a good first move for the alliance because M will not be able to force a win. In both diagrams, there are four available spaces left on Slice 2 after one complete turn.

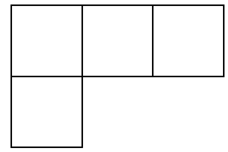
M to play after 1 turn



Slice 2

Diagram 2-17a
The remaining playable spaces on the Slice 2 from Diagram 2-16a.

M to play after 1 turn

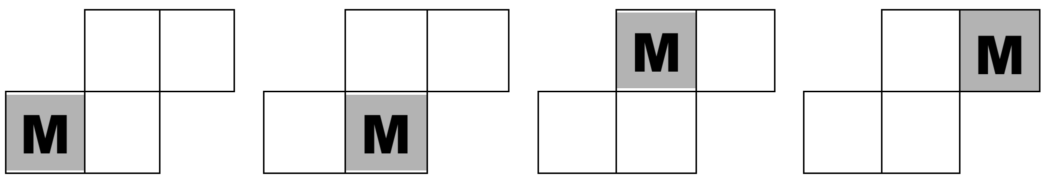


Slice 2

Diagram 2-17b
The remaining playable spaces on the Slice 2 from Diagram 2-16b.

In each game, M has four possible moves in the remaining spaces of Slice 2, which are indicated in the following diagrams.

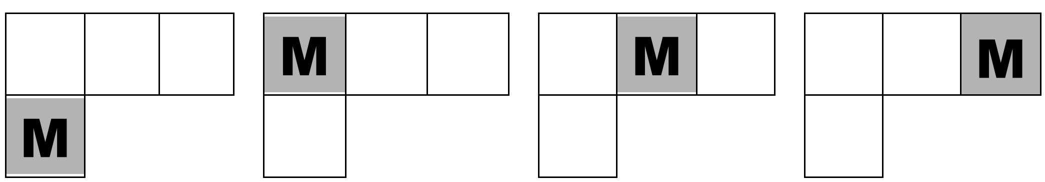
L (or R) to play after 1.5 turns



Slice 2 Slice 2 Slice 2 Slice 2

Diagram 2-18
This shows all four possible places where M can place a piece on Slice 2 from Diagram 2-17a.
The other M piece is on either Slice 1 or Slice 3.

L (or R) to play after 1.5 turns



Slice 2 Slice 2 Slice 2 Slice 2

Diagram 2-19
This shows all four possible places where M can place a piece on Slice 2 from Diagram 2-17b.
The other M piece is on either Slice 1 or Slice 3.

One can see in Diagrams 2-18 and 2-19 that regardless of where M places his next piece, only one alliance member can place a piece on Slice 2 in his next turn. The other must place his piece on either Slice 1 or 3. This means that three more spaces on Slice 2 will be occupied at the end of the second turn, leaving M with only one more space to play.

This partially proves Proposition 2-6, and we will complete this proof later in this section. Until then, we will prove that M will not be able to force a win after playing on a corner or center space using a chart similar to the ones in Diagrams 2-9 and 2-13. This chart shows the number of available spaces left in each slice after each turn in an M-L-R and M-R-L game.

Number of free spaces in each turn of a specific M-L-R & M-R-L game						
Turn Number	M-L-R Game: Who Plays Next	M-R-L Game: Who Plays Next	# of Free Spaces in Slice A	# of Free Spaces in Slice 2	# of Free Spaces in Slice B	Total # of Free Spaces
0	M to play next	M to play next	9	9	9	27
0.5	L to play next	R to play next	8	8	9	25
0.75	R to play next	L to play next	8	6	9	23
1	M to play next	M to play next	8	4	9	21
1.5	L to play next	R to play next	7	3	9	19
1.75	R to play next	L to play next	7	1	9	17
2	M to play next	M to play next	7	1	7	15
2.5	L to play next	R to play next	6	0	7	13

Diagram 2-20

There are a couple of things to note about the chart. First, the first part of the alliance's blocking strategy (L and R covering as much of Slice 2 as possible) is reflected in the moves made in the chart. Second, for the columns "# of Free Spaces in Slice A" and "# of Free Spaces in Slice B", they can either be A = 1 and B = 3 or A = 3 and B = 1. Third, the "Total # of Free Spaces" column can refer to all possible M-L-R and M-R-L games even if the chart refers to one specific game for each turn order. Recall that we demonstrated in the Diagram 2-9 and 2-10 charts that, regardless of where L, R, and M play, each player's piece takes up two spaces.

Diagram 2-20 shows games where the second player is able to move on Slice 2 during the second turn. The following chart shows what happens if the third player is able to move on Slice 2 during the second turn instead.

Number of free spaces in each turn of a specific M-L-R & M-R-L game						
Turn Number	M-L-R Game: Who Plays Next	M-R-L Game: Who Plays Next	# of Free Spaces in Slice A	# of Free Spaces in Slice 2	# of Free Spaces in Slice B	Total # of Free Spaces
0	M to play next	M to play next	9	9	9	27
0.5	L to play next	R to play next	8	8	9	25
0.75	R to play next	L to play next	8	6	9	23
1	M to play next	M to play next	8	4	9	21
1.5	L to play next	R to play next	7	3	9	19
1.75	R to play next	L to play next	7	3	7	17
2	M to play next	M to play next	7	1	7	15
2.5	L to play next	R to play next	6	0	7	13

Diagram 2-21

Although the number of free spaces in Slices A and B deviates in Turns 1.5 and 1.75 from those in Diagram 2-21, the total number of free spaces for each turn in both charts remains the same.

To reiterate what was stated in the last two sections, the best way to prove that M loses in a three-player game is to show that either L or R has at least one more move after M plays his final piece. After two complete turns, there is one space left in Slice 2. After M moves on that space in his last turn, there are thirteen available spaces left in Slices A and B. With thirteen available spaces, it is guaranteed that either L or R will be able to move at least once after M's final turn, which is a loss for M.

This analysis covers all M-L-R and M-R-L games where M plays on either the corner or center square first, so we have partially proved Proposition 2-7. Next, we will analyze all M-L-R and M-R-L games where M's first move is neither corner nor center (the third type of move in Diagram 2-14) to complete the proofs of Propositions 2-6 and 2-7.

L (or R) to play after 0.5 turns

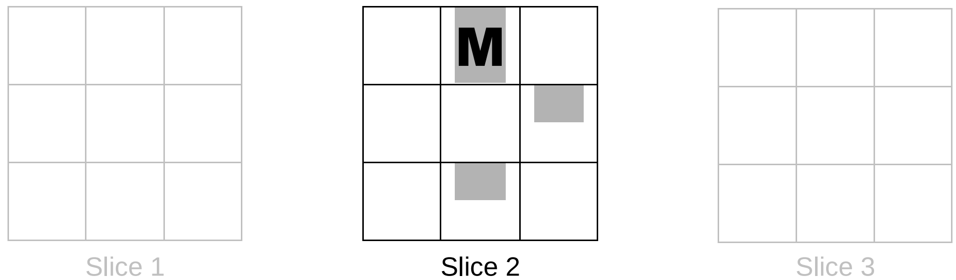


Diagram 2-22
M's first move is on a space that is neither corner nor center.

Diagram 2-23 demonstrates where the alliance can play after M's move on Diagram 2-22. We will show in subsequent examples that these moves are a good first move for the alliance because M will not be able to force a win.

M to play after 1 turn

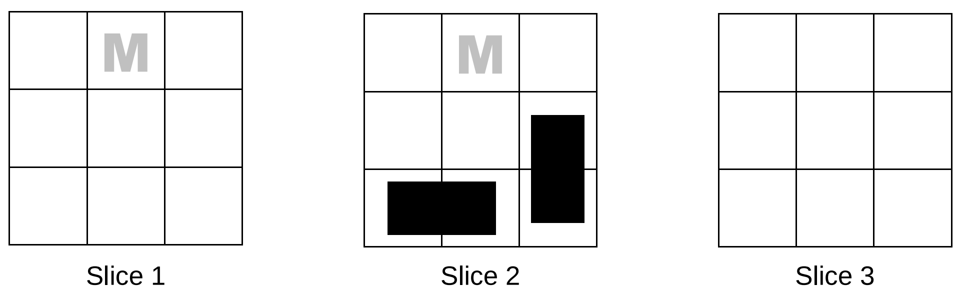


Diagram 2-23
The alliance responds to M's move from Diagram 2-22

There are four spaces left in Diagram 2-23. M has four possible moves in the remaining spaces of Slice 2, which are shaded in the following diagram.

L (or R) to play after 1.5 turns

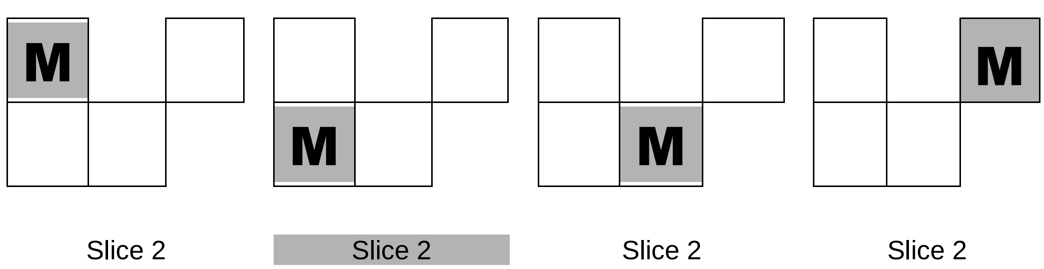


Diagram 2-24
This shows all four possible places where M can place a piece on the Slice 2 from Diagram 2-23.
The other M piece is on either Slice 1 or Slice 3.

One can see that the second possible game marked in Diagram 2-24 prevents the alliance from playing any more pieces on Slice 2. The other three possibilities allow one alliance member to move in Slice 2 in their next turn, leaving M with one move left after two turns. This creates a losing scenario for M similar to the games where M played on a corner or center square (see the charts in Diagrams 2-20 and 2-21). The move in Diagram 2-24's second game is a better move for M than the other games because it leaves M with three available spaces on Slice 2 after two turns instead of one available space. Therefore, we will continue our analysis from that second possible game.

This also concludes the proof of Proposition 2.6.

In our previous examples, we were able to show that the alliance could prevent M from winning without having to do the second part of their blocking strategy (playing pieces above and below available M spaces). However, now that M can have three available spaces in Slice 2 after 1.5 turns, the alliance will have to execute the second part of their blocking strategy to prevent M from winning. We will demonstrate this in subsequent examples.

L (or R) to play after 1.5 turns

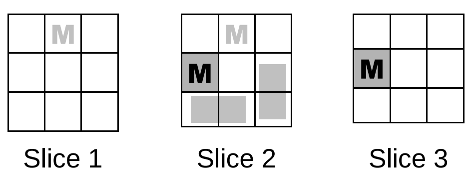


Diagram 2-25a
M plays second piece on Slices 2 and 3 from Diagram 2-24's second possible game.

L (or R) to play after 1.5 turns

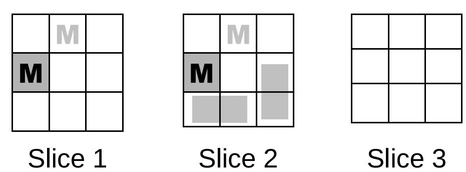


Diagram 2-25b
M plays second piece on Slices 1 and 2 from Diagram 2-24's second possible game.

From Diagram 2-24's second game, there are two possible ways M can place his second piece—the piece can either traverse Slices 2 and 3 (see Diagram 2-25a) or it can traverse Slices 1 and 2 (see Diagram 2-25b).

Note: If we can show that the alliance can prevent M from winning in these two cases, we will have finished considering all possible M moves and prove Proposition 2.7.

Let's examine how each of these games continue, starting with Diagram 2-25a.

L (or R) to play after 1.5 turns

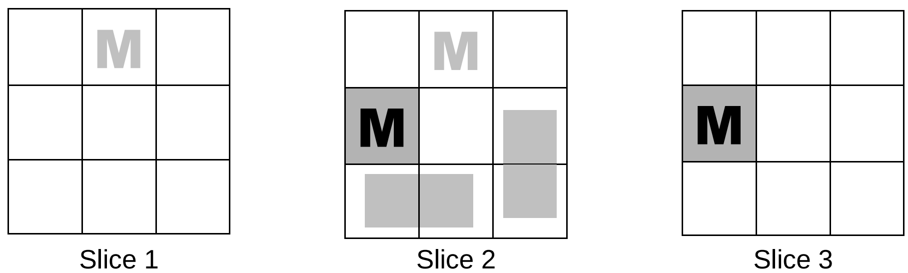


Diagram 2-26
From Diagram 2-25a, M played his second piece across Slices 2 & 3

M has three available spaces to play on Slice 2 after playing his second piece. However, the alliance can perform the second half of their blocking strategy and block one of those spaces by playing on squares above and below an available space.

M to play after 2 turns

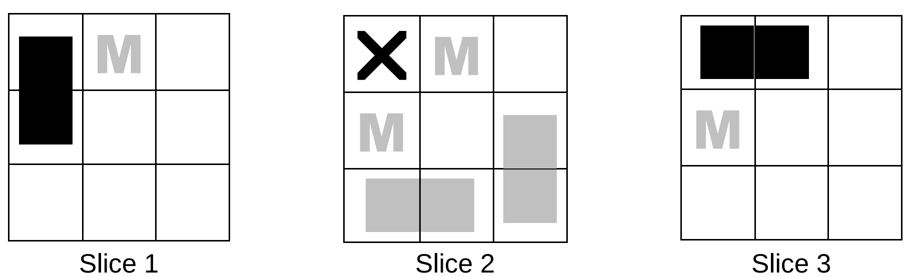


Diagram 2-27
The ‘X’ indicates that M can no longer play on that square because
L and R blocked M by playing pieces above and below the square.

The alliance has blocked M from playing on the upper-left corner of Slice 2 by playing their pieces on the upper-left corner of Slices 1 & 3 during their turn. This now leaves M with two available spaces in Slice 2 instead of three. M has four possible moves left, but they can be reduced to one.

L (or R) to play after 2.5 turns

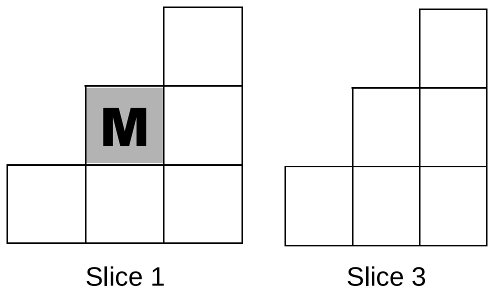


Diagram 2-28a
From Diagram 2-27, M plays on the center square.
It also shows the remaining spaces for L & R.

L (or R) to play after 2.5 turns

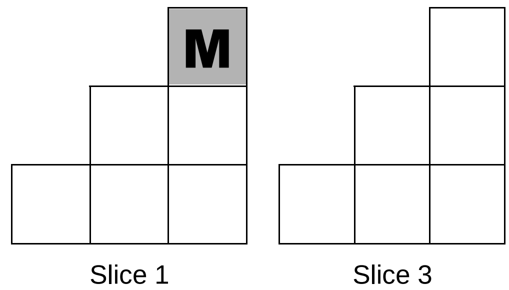


Diagram 2-28b
From Diagram 2-27, M plays on the upper-right square.
It also shows the remaining spaces for L & R.

Diagrams 2-28a and 2-28b show two possible games where M plays on one of the available squares on Slice 2. In both diagrams, M's piece traverses Slices 1 and 2. Note that the remaining available spaces for M to play in Diagram 2-27 are the same in Slices 1 and 3. Thus, M playing on Slices 2 and 3 is the same as playing on Slices 1 and 2, which is why we only show two examples in Diagram 2-28 instead of four.

In Diagram 2-28a, one can see that after M plays on the center square, M still has one move left. This is because the alliance is unable to block the upper-right corner of Slice 2. It is impossible for R to play on the upper-right corner of Slice 1 or 3. On the other hand, if M plays on the upper-right corner as shown in Diagram 2-28b, then the alliance can block M's final move in Slice 2's center square by placing their pieces over the center square of Slices 1 and 3, thus eliminating M from the game. Therefore, the better move for M is to play on the center square.

L (or R) to play after 2.5 turns

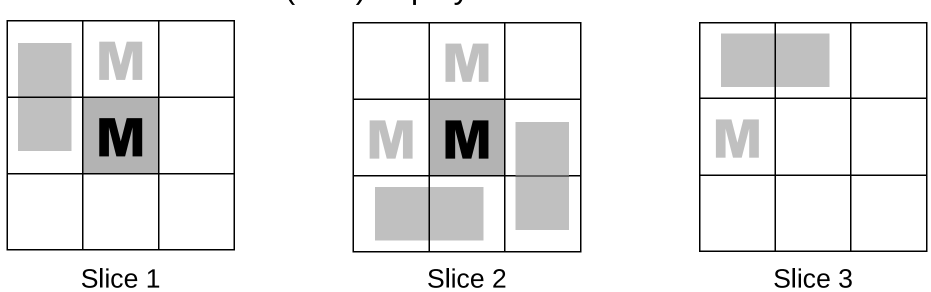


Diagram 2-29
From Diagram 2-28a,
M has played his second piece across Slices 1 & 2 on the center square

Since M playing on Slices 2 and 3 is symmetrical to M playing on Slices 1 and 2, there is no need to show the game where M plays on Slices 2 and 3 in our subsequent examples. M now has only one space left to play on Slice 2.

M to play after 3 turns

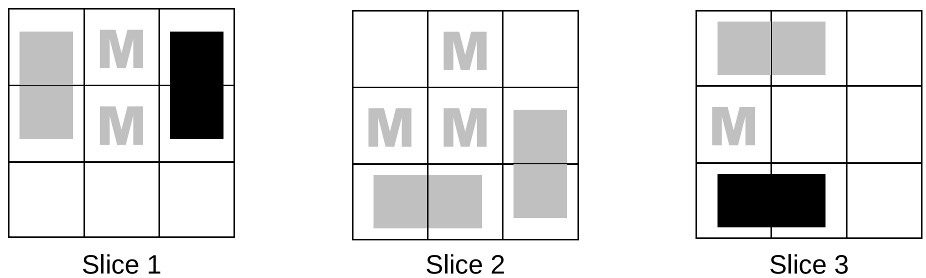


Diagram 2-30
The alliance responds to M's move in Diagram 2-29.

The alliance is unable to block M from playing on the upper-right corner, but L will force M to play on only Slices 2 and 3 by placing his piece over the upper-right corner of Slice 1.

L (or R) to play after 3.5 turns

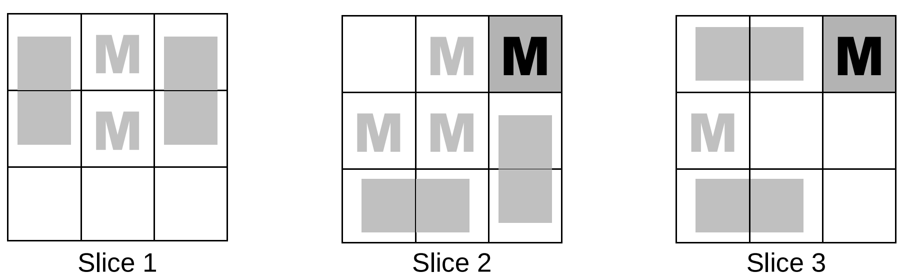


Diagram 2-31
M plays his final piece.

One can see that after M plays his final piece on the upper-right corner, both L and R can still play on the remaining available spaces. This is a loss for M because the alliance has at least one legal move left in the combined six available spaces of both Slices 1 and 3 after M plays his final piece. The previous examples apply to both M-L-R and M-R-L turn-order games. We have covered the game from Diagram 2-25a, so let's examine the game from Diagram 2-25b.

L (or R) to play after 1.5 turns

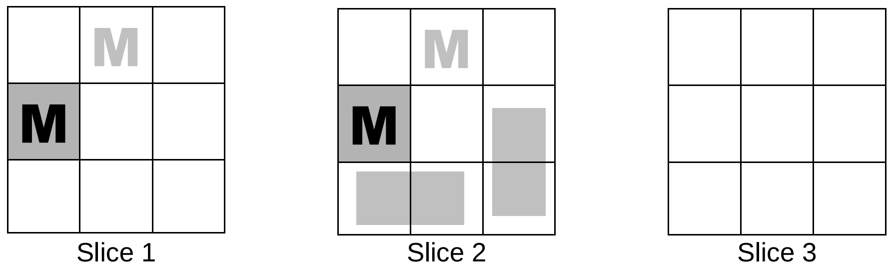


Diagram 2-32
From Diagram 2-25b, M has played his second piece across Slices 1 & 2.

M has three available spaces to play in Slice 2, so the alliance will perform the second part of their blocking strategy by playing on space above and below an available M space. However, M has placed his two pieces in such a way that L and R cannot to block the upper-left corner of Slice 2 at any point during the game. This tactic is called *reserving a space*, and places M in a better position than the game in Diagram 2-25a because M has guaranteed that one of his moves will not be blocked. However, the alliance is still capable of blocking M's other spaces.

M to play after 2 turns

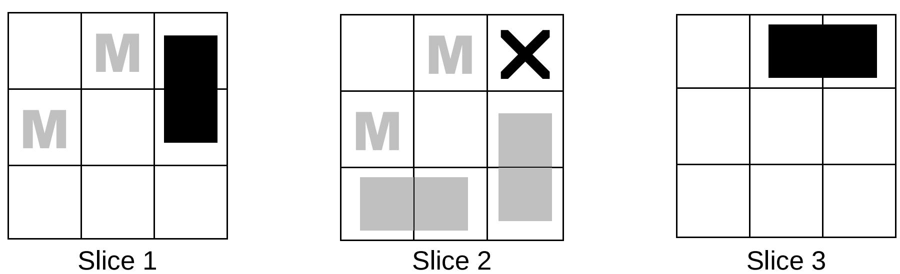
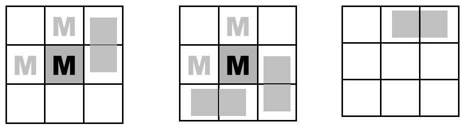


Diagram 2-33
The 'X' indicates that M can no longer play on that square because
L and R blocked M by playing pieces above and below the square.

When L and R covered the upper-right corner of Slices 1 and 3, it removes the upper-right corner of Slice 2 as a playable space for M. This leaves M with two more available moves on Slice 2. If M were to play on the reserved upper-left corner of Slice 2, then the alliance can cooperatively block the spaces above and below M's last available move in Slices 2's center space, which would result in a loss for M. However, the alliance cannot block the upper-left corner of Slice 2 because it is reserved for M. Therefore, it is a better move for M to play on the center square because M will still have one more move after he plays on the center square.

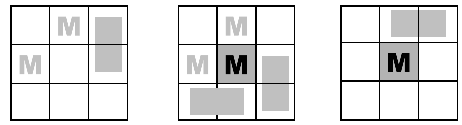
L (or R) to play after 2.5 turns



Slice 1 Slice 2 Slice 3

Diagram 2-34a
M plays on the unprotected center square.

L (or R) to play after 2.5 turns

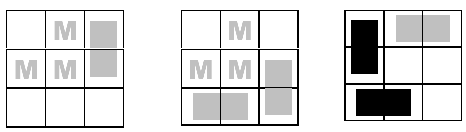


Slice 1 Slice 2 Slice 3

Diagram 2-34b
M plays on the unprotected center square.

Diagram 2-34a shows M playing a piece on the center square that traverses Slices 1 and 2. Diagram 2-34b shows M playing a piece on the center square that traverses Slices 2 and 3. Our following examples will follow these two games.

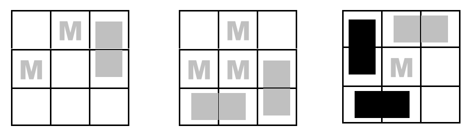
M to play after 3 turns



Slice 1 Slice 2 Slice 3

Diagram 2-35a
The alliance responds to
M's move in Diagram 2-34a.

M to play after 3 turns

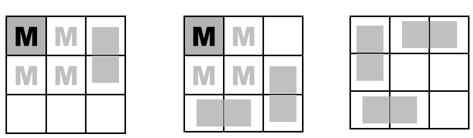


Slice 1 Slice 2 Slice 3

Diagram 2-35b
The alliance responds to
M's move in Diagram 2-34b.

After M's third turn, L and R place their pieces on Slice 3. In both Diagrams 2-35a and 2-35b, L covering the upper-left corner of Slice 3 forces M to play his last piece on the upper-left corner of Slices 1 & 2.

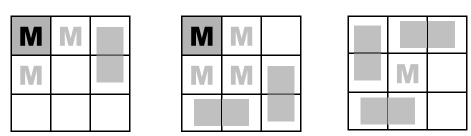
L (or R) to play after 3.5 turns



Slice 1 Slice 2 Slice 3

Diagram 2-36a
M played his final move.

L (or R) to play after 3.5 turns



Slice 1 Slice 2 Slice 3

Diagram 2-36b:
M played his final move;

In both Diagrams 2-36a and 2-36b, one can see that after M plays his final piece, there are still six available spaces that the alliance can play on. Since the alliance is still able to move after M plays his final piece, then M has lost the game.

This concludes our analysis of M-L-R and M-R-L games where M plays on a space that is neither corner nor center first.

This also concludes the proof of Proposition 2-7.

This completes our coverage of all possible 3x3x3 Domineering games. We have proved that regardless of whether M plays first, second, or third, M can never force a win in a 3x3x3 Domineering game if L and R collude to stop him using the strategies we described in this chapter.

This completes the proof of Proposition 2-1.

In other words, a 3x3x3 Domineering game is in the outcome class $\mathcal{Q}$.

3.0 – Strategies to Solve a 4x4x3 Domineering Game: Part 1

In Chapters 3 and 4, we are going to extend the strategies we defined in the previous chapter to solve a 4x4x3 Domineering game. Since a 4x4x3 game is not symmetric on all sides, this affects our analysis of where L, R, and M play in that game's board. We mentioned in the first chapter that the edges of a 3D Domineering board are parallel to the x-, y-, and z-axes. For this thesis, we will define where L, R, and M play along those axes.

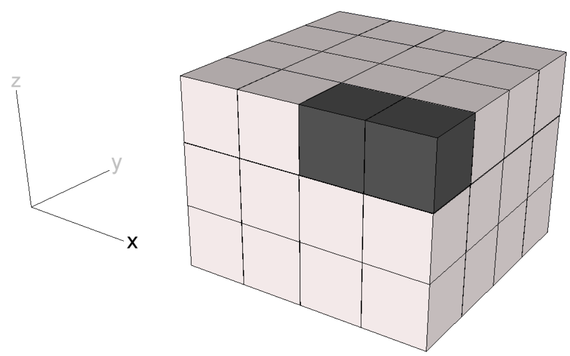


Diagram 3-1
A 4x4x3 board aligned with the x, y, and z axes.
R's move (highlighted in dark gray) is parallel to the x-axis.

In Diagram 3-1, we establish that R's moves will be parallel to the x-axis (example highlighted in dark gray). In this thesis, we consider the *A* component of an *A* x *B* x *C* game to be parallel to the x-axis. Therefore, R will be playing on the first 4 component of a 4x4x3 game according to our definition, regardless of turn-order.

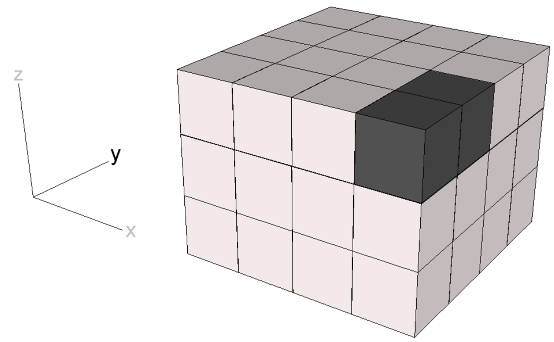


Diagram 3-2
A 4x4x3 board aligned with the x, y, and z axes.
L's move (highlighted in dark gray) is parallel to the y-axis.

In Diagram 3-2, we establish that L's moves will be parallel to the y-axis (example highlighted in dark gray). In this thesis, we consider the *B* component of an *A* x *B* x *C* game to be parallel to the y-axis. Therefore, L will be playing on the second 4 component of a 4x4x3 game according to our definition, regardless of turn-order.

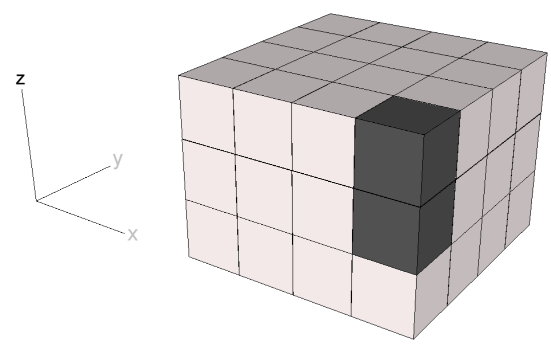


Diagram 3-3
A 4x4x3 board aligned with the x, y, and z axes.
M's move (highlighted in dark gray) is parallel to the z-axis.

In Diagram 3-3, we establish that M's moves will be parallel to the z-axis (example highlighted in dark gray). In this thesis, we consider the *C* component of an *A* x *B* x *C* game to be parallel to the z-axis. Therefore, M will be playing on the 3 component of a 4x4x3 game according to our definition, regardless of turn-order.

We established these definitions to make it clear where M is playing when presenting the proof for solving a 4x4x3 game. In Chapter 3, we will show that the alliance can prevent M from winning a 4x4x3 Domineering game if M plays on the 3 component. In Chapter

4, we will show that the alliance can prevent M from winning a 4x4x3 Domineering game if M plays on the second 4 component (which will be subsequently be referred to as a 4x3x4 game in this thesis).

In this chapter, we are going to prove the following:

Proposition 3-1: *M is unable to force a win in a 4x4x3 game, where M plays on the 3 component, if the L & R alliance teams up against him.*

The strategies that L & R will use to do this are similar to the strategies that prevented M from winning in a 3x3x3 board. Those strategies are to cover up as much of Slice 2 as possible to limit the number of available moves M has on the board. Then the alliance covers the spaces above and below any remaining spaces in Slice 2 to eliminate M's remaining moves.

To start the proof of Proposition 3.1, we will prove the following for 4x4x3 Domineering games where M plays on the 3 component:

Proposition 3-2: *Regardless of turn order, when the alliance covers as much of Slice 2 as possible, the maximum number of available spaces M can play on Slice 2 is three, assuming best play from M and the alliance.*

3.1 – Analysis of L-R-M and R-L-M Games

To prove Proposition 3-2, we will prove the following for 4x4x3 Domineering games where M plays on the 3 component:

Proposition 3-3: *In L-R-M and R-L-M turn order games, the alliance can limit the number of M's available spaces in Slice 2 to a maximum of two after 2.5 turns.*

Let's examine how the alliance can accomplish that.

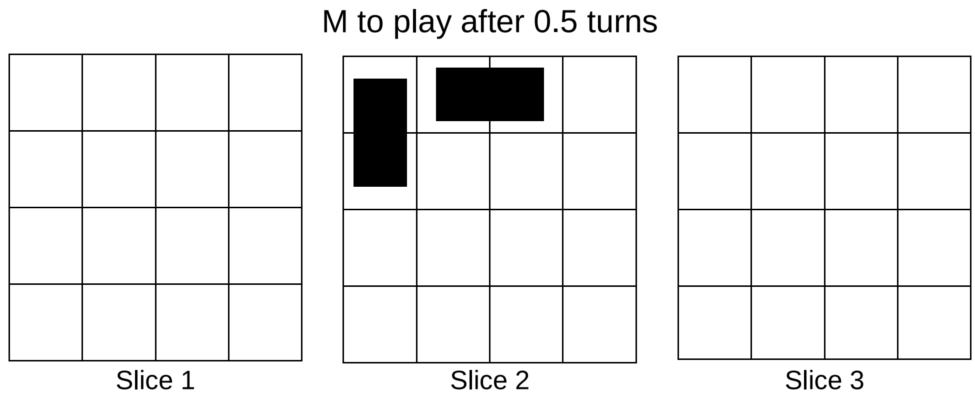


Diagram 3-4
The alliance plays on the upper-left corner of Slice 2.

In Diagram 3-4, the alliance played on the upper-left corner of Slice 2. By the end of this chapter, we will show that these are a good first move for the alliance because M will not be able to force a win. M's next move will cover one of the twelve remaining spaces in Slice 2, so our next examples will cover all of those twelve possibilities, starting with M playing on Row 2, Column 4 in Diagram 3-2.

Diagram 3-5 shows M's piece traversing Slices 1 and 2, but in this and subsequent sections (3.2 and 3.3), it does not matter if M's piece traverses Slices 1 and 2 or Slices 2 and 3. We are more concerned with how Slice 2 is populated in the first turns of a 4x4x3 game. Nonetheless, we will address how the endgame of a 4x4x3 game is affected by whether a piece is placed on Slices 1 and 2 or Slices 2 and 3 in Section 3.4.

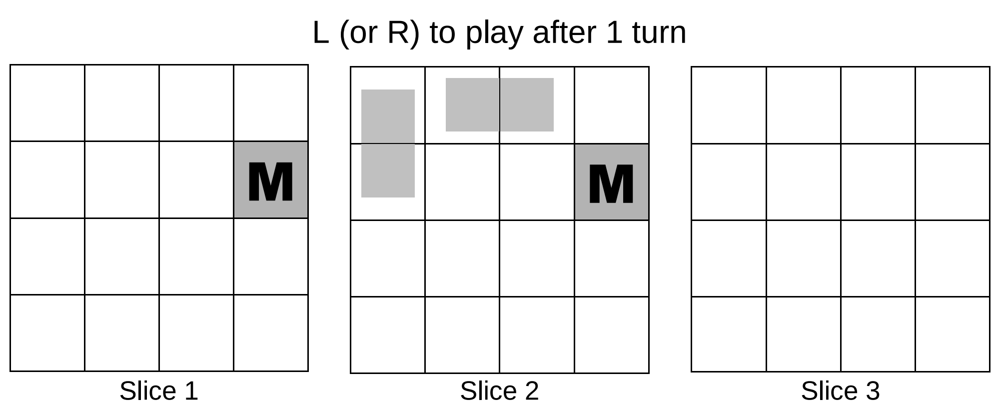


Diagram 3-5
M's move on Slice 2 prevents L & R from playing on the upper-right corner of the board.

This appears to be a good move for M because he reserved a space (meaning that the alliance cannot place a piece in Slice 2 to cover it). However, this situation is okay for the alliance because it will limit M to reserve only one more space in Slice 2.

M to play after 1.5 turns

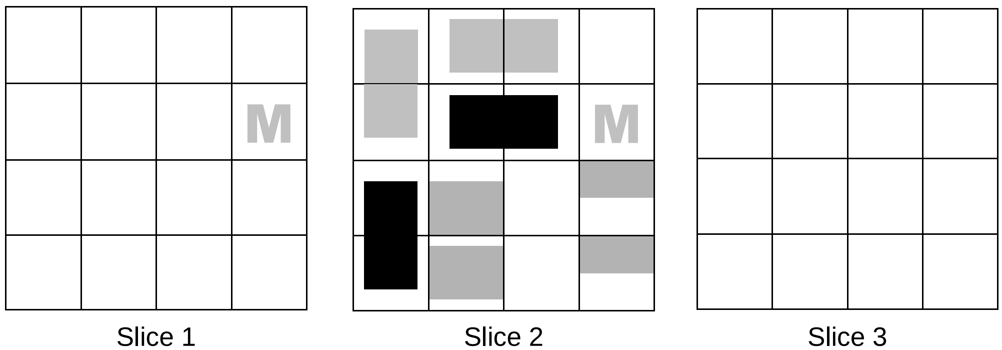


Diagram 3-6
From Diagram 3-5, L & R play pieces that create a 2x3 rectangular area (shaded above)

In Diagram 3-6, L & R have placed pieces that create a 2x3 rectangular area in Slice 2, which is shaded in the above diagram. This area allows both alliance members to play on Slice 2 in their next turn regardless of where M plays next.

M to play after 2.5 turns on each board

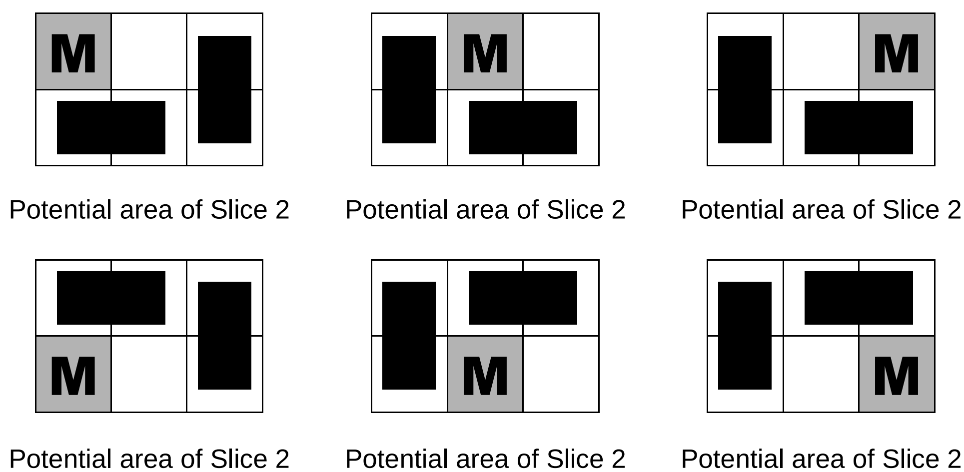


Diagram 3-7
Six possible outcomes for the shaded squares in Diagram 3-6.

Note that in each of the above cases, M is left with just one available space in the 2x3 rectangular area. That one available space and the reserved space in the upper-right corner of Diagram 3-6 show that the alliance is able to restrict M to two available spaces on Slice 2 after 2.5 turns. However, this example only applies to M playing on Row 2, Column 4 as his first move. We'll next review M playing on Row 2, Column 2 and Row 1, Column 4 in the following diagrams.

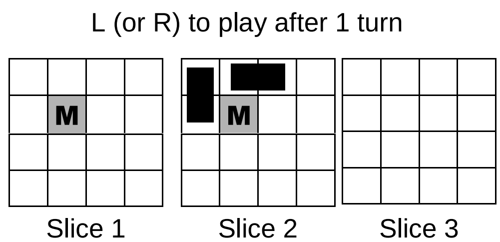


Diagram 3-8a
After Diagram 3-4,
M plays on Row 2, Column 2

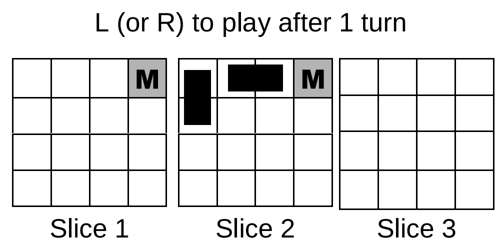


Diagram 3-8b
After Diagram 3-4,
M plays on the upper-right corner

The next diagrams show how the alliance can respond to the moves M made in Diagrams 3-8a and 3-8b.

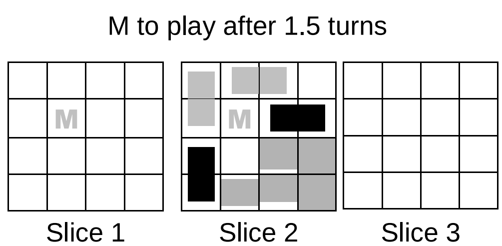


Diagram 3-9a
After Diagram 3-8a,
L & R play pieces to create a
2x3 rectangular area (shaded above)

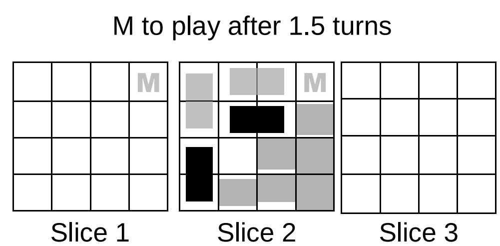


Diagram 3-9b
After Diagram 3-8b,
L & R play pieces to create a
2x3 rectangular area + 1 (shaded above)

In Diagram 3-9a, although the alliance helped M reserve a spot in the upper-right corner, their moves created a 2x3 rectangular area (shaded in Diagram 3-9a). We have already demonstrated in Diagram 3-7 that this area will permit M to reserve only one more space, as both alliance members can move in Slice 2 in their next turn. This leaves M with a total of two available spaces in Slice 2 after 2.5 turns.

In Diagram 3-9b, the alliance created a 2x3 rectangular area with an additional space appended to it (shaded in Diagram 3-9b). If we treat this area like the 2x3 rectangular area in Diagram 3-9a and presume that the appended space is a reserved space for M, then we will arrive at the same conclusion as Diagram 3-9a—the alliance has left M with only two available spaces in Slice 2 after 2.5 turns.

The next diagrams demonstrate what happens if M plays on Row 2, Column 3 as his first move. This covers the fourth possible staring move for M after Diagram 3-4.

M to play after 1.5 turns

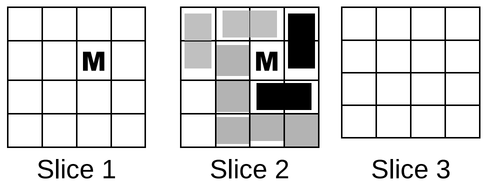


Diagram 3-10a
After Diagram 3-4,
M plays on Col 3, Row 2
L & R respond and created
seven contiguous spaces (shaded above)

M to play after 2.5 turns on each board

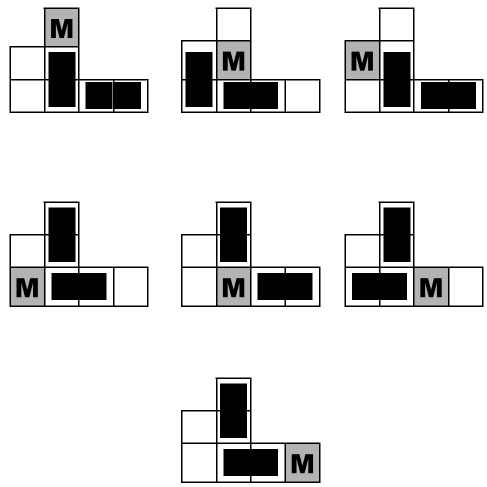


Diagram 3-10b
Regardless of where M plays after
Diagram 3-10a, there will always be left
two spaces left in Slice 2.

If M plays on Column 3, Row 2 as shown in Diagram 3-10a, the alliance can create an area of seven contiguous spaces in their next move (shaded in Diagram 3-10a). As indicated in Diagram 3-10b, the alliance members are both able to move in their next turn regardless of where M plays on that shaded area, leaving M with two available spaces in Slice 2 after 2.5 turns.

So far, we have demonstrated in Diagrams 3-6 to 3-10b that the alliance can hold M to two available spaces in Slice 2 after 2.5 turns if M were to play on any of the top-half squares of Slice 2 in his first turn.

This leaves eight more possible moves for M to play in his first turn. Those remaining moves are on the lower half of Slice 2, and regardless of where M plays on that area, the alliance will always be able to create a 2x4 rectangular area minus one square by playing on the top half of Slices 2, as shown in the following diagram.

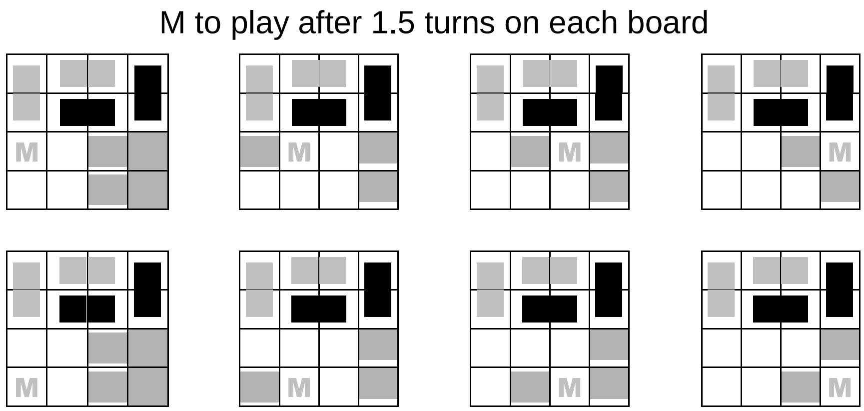


Diagram 3-11
Demonstrates eight possible moves for M on Slice 2's lower half.
L & R playing on top half creates 2x4 area minus one area (shaded).

The reader can verify in Diagram 3-11 that regardless of where M plays next in the shaded area of all eight possible games, the alliance members will both be able to play on Slice 2 in their next turns. This leaves M with two available spaces in Slice 2 after 2.5 turns.

We have exhausted all possible moves for M to play on his first turn after the alliance's first moves. We have shown that for the twelve possible first-turn M moves on Slice 2, the alliance will leave M with at most two available spaces in Slice 2 in L-R-M and R-L-M turn-order games after 2.5 turns.

This concludes the proof of Proposition 3-3.

3.2 – Analysis of M-L-R and M-R-L Games

In this section, we will continue the proofs of Propositions 3-1 and 3-2 by proving the following for 4x4x3 Domineering games where M plays on the 3 component:

Proposition 3-4: *In M-L-R and M-R-L turn order games, the alliance can limit the number of M's available spaces in Slice 2 to a maximum of three after 3 turns.*

There are sixteen starting moves for M in an M-L-R and M-R-L game. However, we can break this down to three types of moves thanks to symmetry, as demonstrated in Diagram 3-12.

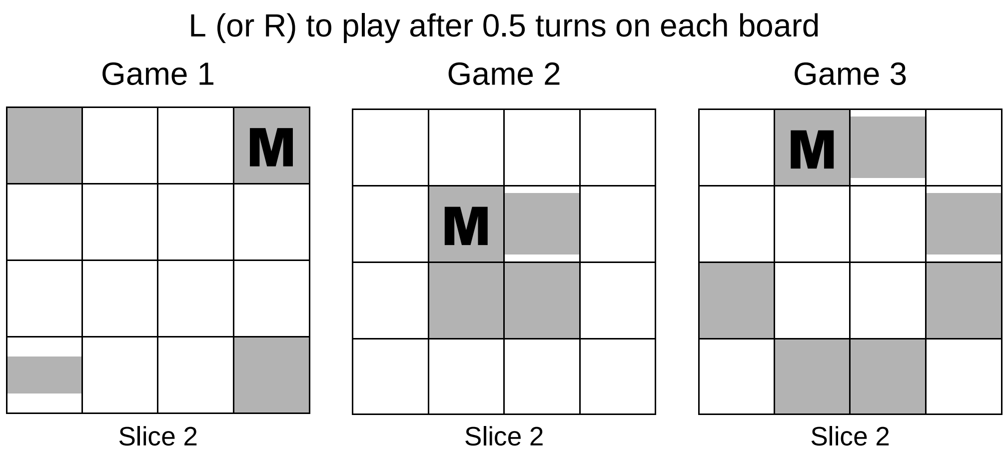


Diagram 3-12
Taking symmetry into account, M has three starting moves in a M-L-R (or M-R-L) game. Symmetric squares are shaded.

The first board of Diagram 3-12 (Game 1) represents M playing on any of the corner spaces (which are shaded). The second board (Game 2) represents M playing on one of the four center squares of Slice 2 (which are shaded). The third board (Game 3) represents M playing on spaces that are neither corner nor center (which are shaded).

In order for the alliance to restrict M to a maximum of three reserved spaces after two turns, they have to be careful not to allow M to reserve two spaces before M's second turn and three spaces before M's third turn.

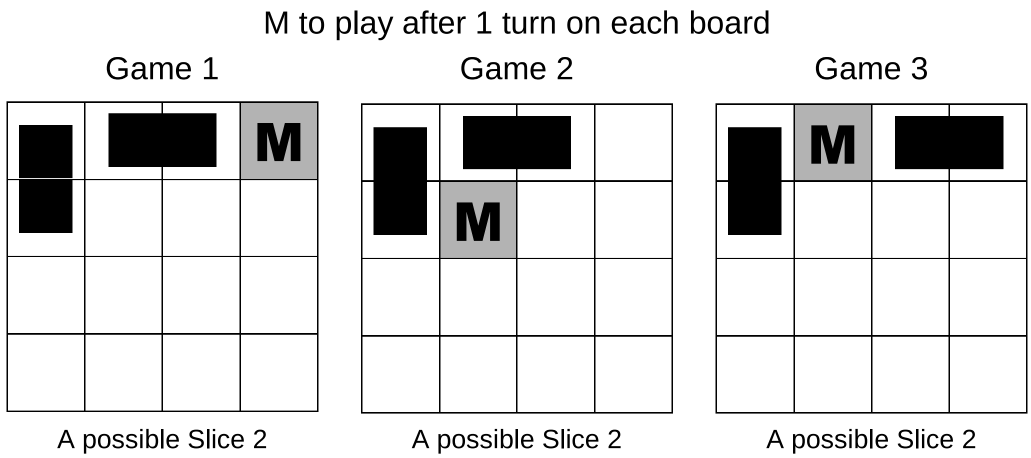


Diagram 3-13

From Diagram 3-12, these diagrams show where the alliance should play after M's first move.

Diagram 3-13 shows how the alliance should respond to the three types of starting moves M can make. By the end of this chapter, we will show that these are a good first move for the alliance because M will not be able to force a win. In Games 1 and 3, the reader can see that the alliance has not left M any openings to reserve a space in his second turn. However, M can reserve a space in Game 2 by playing on Row 2, Column 4 in his second turn. M playing on one of the center four squares is actually M's best move, because regardless of where the alliance places their pieces in their next turns, M will always be able to reserve a space in his second turn. However, because we will prove that the alliance's initial move in Diagram 3-13's Game 2 is a win for the alliance, the analysis of all other options for L & R in Game 2 is not necessary.

For the purpose of proving that the alliance can hold M to a maximum of three reserved spaces, let's examine all three possible starting moves for M, beginning with M playing on a corner space (Game 1).

M to play after 2 turns on each board (all represent Slice 2)

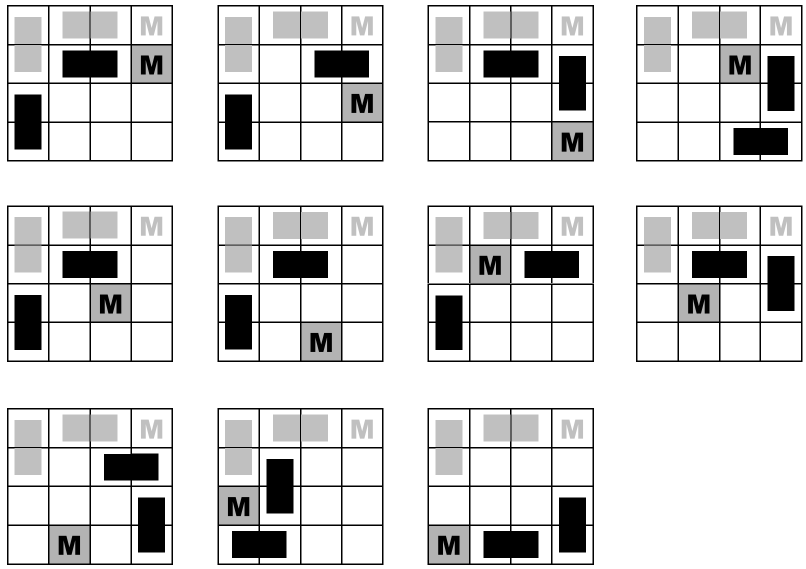


Diagram 3-14
These boards showcase the eleven possible moves M can play after Diagram 3-13's Game 1 (which are shaded). Also shown is the alliance's response to each shaded move.

Diagram 3-14 shows all eleven possible moves that M can make from Diagram 3-13's Game 1 configuration. Also shown is how the alliance should respond to those shaded moves in each board.

Note that after two turns, there are six spaces left in every board and no reserved moves yet for M. We will examine those spaces after we cover the possible board configurations for Diagram 3-13's Games 2 and 3.

Now let's examine the possible moves for M when he plays on one of the center squares (Diagram 3-13's Game 2):

M to play after 2 turns on each board (all represent Slice 2)

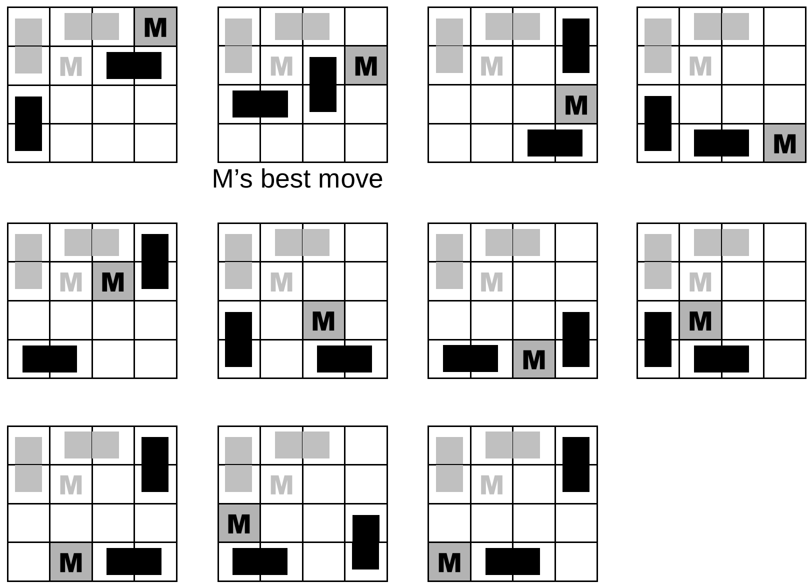


Diagram 3-15
These boards showcase the eleven possible moves M can play after Diagram 3-13’s Game 2 (which are shaded). Also shown is the alliance’s response to each shaded move.

Diagram 3-15 shows all eleven possible moves that M can make from Diagram 3-13’s Game 2. Also shown is how the alliance should respond to those shaded moves in each board.

Note that after two turns, there are six spaces left in every board. The board marked “M’s best move” is indeed M’s best move because it is the only one that reserves a space for M, leaving the alliance at a disadvantage at the start of Turn 2. We’ll analyze the remaining spaces from Diagrams 3-14 and 3-15 after examining the possible moves from Diagram 3-13’s Game 3 scenario.

M to play after 2 turns on each board (all represent Slice 2)

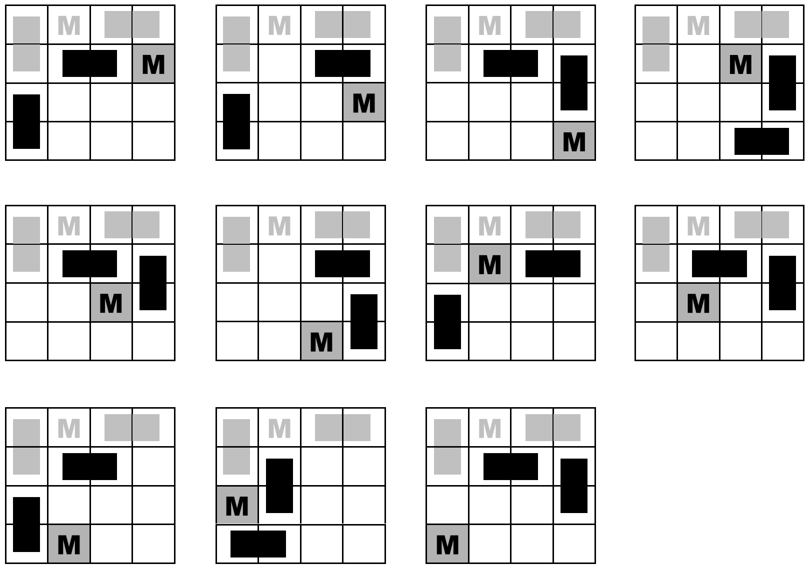


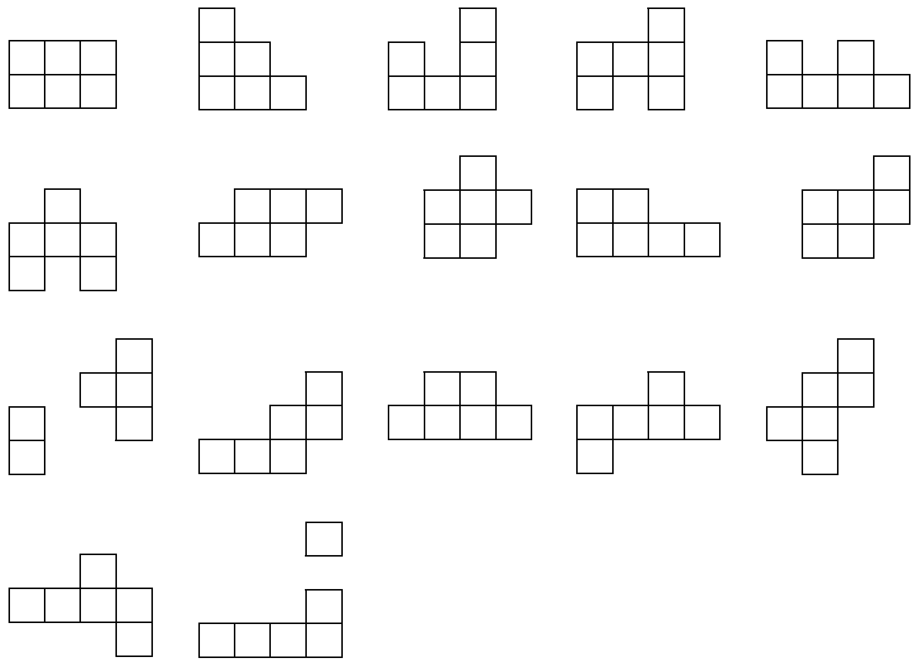
Diagram 3-16

These boards showcase the eleven possible moves M can play after Diagram 3-13's Game 3 (which are shaded). Also shown is the alliance's response to each shaded move.

Diagram 3-16 shows all eleven possible moves that M can make from Diagram 3-13's Game 3 configuration. Also shown is how the alliance should respond to those shaded moves in each board.

Note that after two turns, there are six spaces left in every board. Some of the remaining spaces in this diagram are similar to remaining spaces in Diagrams 3-14 and 3-15. Removing symmetrical cases, the following is a collection of remaining spaces from Diagrams 3-14 to 3-16.

M to play after 2 turns on each board



M’s best move

Diagram 3-17
The above are all six-space board configurations from Diagrams 3-14 to 3-16
(not counting symmetry)

We have determined in Diagram 3-7 that the 2x3 rectangular board (the upper-left board of Diagram 3-17) allows both L & R to move in Slice 2 on their turns after M, leaving M with just one available square in that area. This leaves M with only one available space left in Slice 2 after three turns, so M should avoid placing pieces that will help the alliance create the 2x3 rectangular area.

The reader can verify that for all the other board configurations, there exists at least one M move that allows only one alliance member to move in Slice 2 on his next turn, forcing the other alliance member to play on either Slice 1 or Slice 3. Thus, this leaves M with three available spaces in Slice 2 after three turns.

This completes the proof of Proposition 3-4.

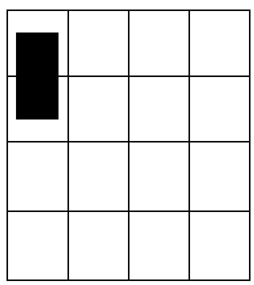
3.3 – Analysis of L-M-R and R-M-L Games

In this section, we continue the proofs of Propositions 3-1 and 3-1 by proving the following for 4x4x3 Domineering games where M plays on the 3 component:

Proposition 3-5: *In L-M-R and R-M-L turn order games, the alliance can limit the number of M's available spaces in Slice 2 to a maximum of three after 2.75 turns.*

Diagrams 3-18a and 3-18b show examples of L and R's first moves in an L-M-R and an R-M-L game respectively.

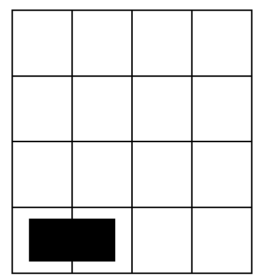
M to play after 0.25 turns



Slice 2 (L-M-R game)

Diagram 3-18a
L's first move

M to play after 0.25 turns



Slice 2 (R-M-L game)

Diagram 3-18b
R's first move

By the end of this chapter, we will show that these are a good first move for the alliance because M will not be able to force a win. The following diagrams in this section will follow an L-M-R game, but they can also correlate to a R-M-L game because of rotational symmetry (the reader can confirm this by looking at Diagrams 3-18a and 3-18b).

There are fourteen possible moves for M to play in his next turn, but we can reduce this analysis to seven combinations.

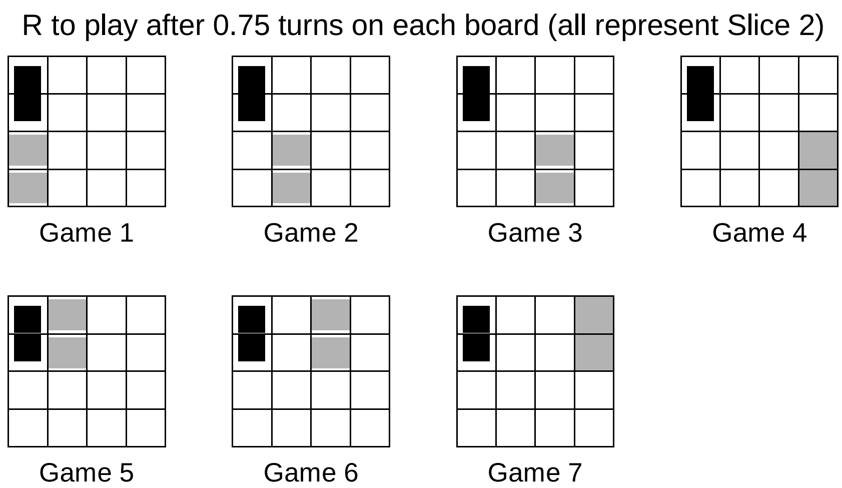


Diagram 3-19
The shaded squares in each diagram demonstrate potential spaces where M can play.

Each Slice 2 in Diagram 3-19 has two shaded squares. These shaded squares represent spaces where M can play in his next turn. The reason they are shaded is because L and R are going to agree not to play on those spaces, essentially playing “one-hand tied”. We will show that the alliance can still prevent M from winning using this self-imposed handicap. Note that because the alliance has agreed not to play on the shaded squares, we will consider one of the two shaded squares as a reserved space for M.

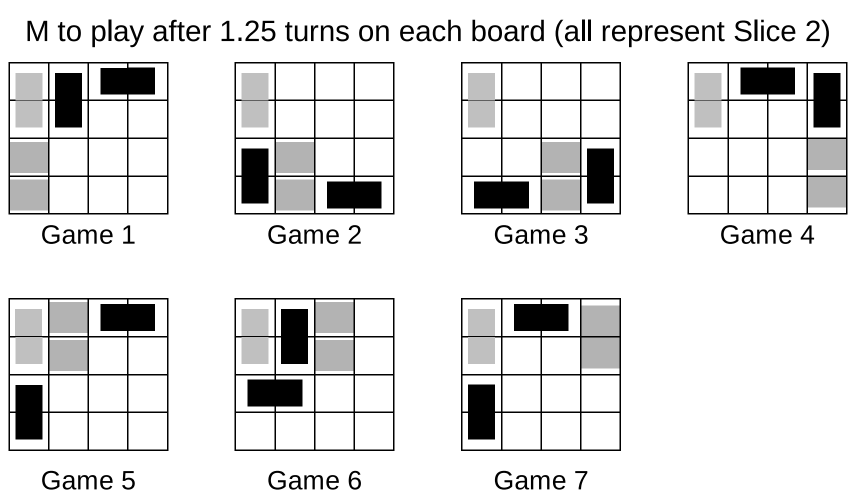


Diagram 3-20
From Diagram 3-19, the alliance members make their next move,
agreeing not to play on the shaded squares of each game.

Each game in Diagram 3-20 shows how the alliance can react to M’s move in either shaded square. We will show that all the moves made in each game are a winning strategy for L and R. Observe in the seven games that there are three possible board configurations left on Slice 2 (excluding symmetry).

M to play after 1.25 turns on each board configuration

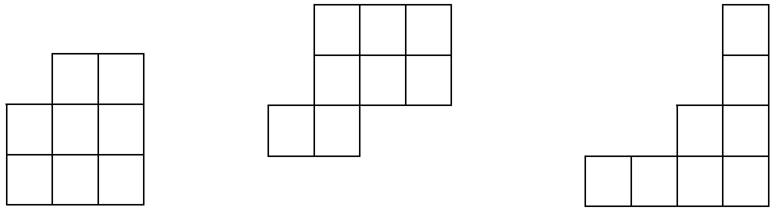


Diagram 3-21a **Diagram 3-21b** **Diagram 3-21c**

These diagrams are the remaining spaces from the seven games in Diagram 3-20 (excluding symmetry)

We will show that for all possible board configurations in Diagrams 3-21a, 3-21b, and 3-21c, M can additionally reserve a maximum of two spaces after 2.75 turns. Let's start with Diagram 3-21a.

M to play after 2.25 turns on each board (all represent Slice 2)

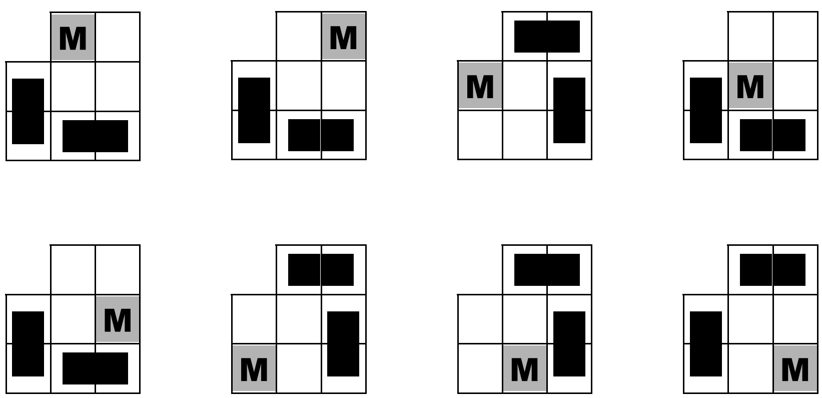


Diagram 3-22
These boards showcase the eight possible moves M can play (shaded in the above diagrams) after Diagram 3-21a.
Also shown is the alliance's response to each shaded move.

Diagram 3-22 shows all eight possible moves that M can make from Diagram 3-21a. Also shown is how the alliance should respond to those shaded moves in each board (we will show later in this chapter that the alliance's response in each diagram is a winning strategy for the alliance).

Note that after 2.25 turns, there are three spaces left in every board with M to play next. If M were to play on the corner of the remaining spaces on each board, it would give M two reserved spaces and prevent the alliance from placing any more pieces on Slice 2. We established earlier that one of the shaded squares in Diagram 3-20 was a reserved space for M, so the maximum number of reserved spaces for M in Diagram 3-21a is three.

Let's now examine the possible moves that can be made in Diagram 3-21b.

M to play after 2.25 turns on each board (all represent Slice 2)

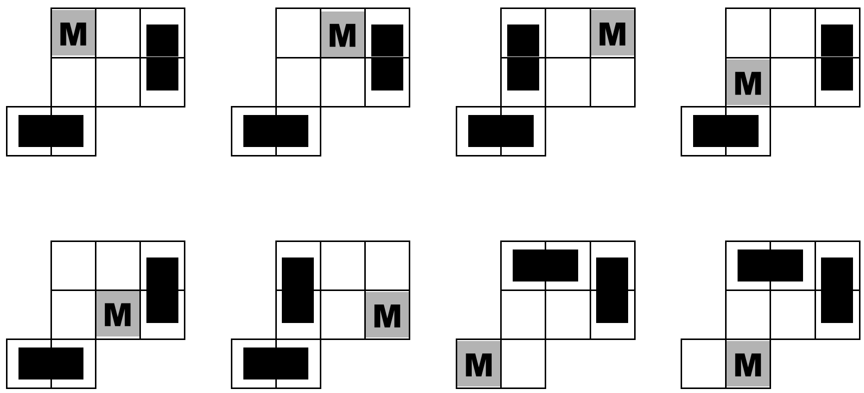


Diagram 3-23
These boards showcase the eight possible moves M can play
(shaded in the above diagrams) from Diagram 3-21b.
Also shown is the alliance's response to each shaded move.

Diagram 3-23 shows all eight possible moves that M can make from Diagram 3-21b. Also shown is how the alliance should respond to those shaded moves in each board (we will show later in this chapter that the alliance's response in each diagram is a winning strategy for the alliance).

Note that after 2.25 turns, there are three spaces left in every board with M to play next. In the first seven board configurations of Diagram 3-23, if M were to play on the corner of the remaining spaces on each board, it would give M two reserved spaces and prevent the alliance from placing any more pieces on Slice 2. In the eighth board of Diagram 3-23, if M's next move is not to the left of the shaded M square, then it would give M two reserved spaces and prevent the alliance from placing any more pieces on Slice 2. We established earlier that one of the shaded squares in Diagram 3-20 was a reserved space for M, so the maximum number of reserved spaces for M in Diagram 3-21b is three.

Let’s now examine the possible moves that can be made in Diagram 3-21c.

M to play after 2.25 turns on each board (all represent Slice 2)

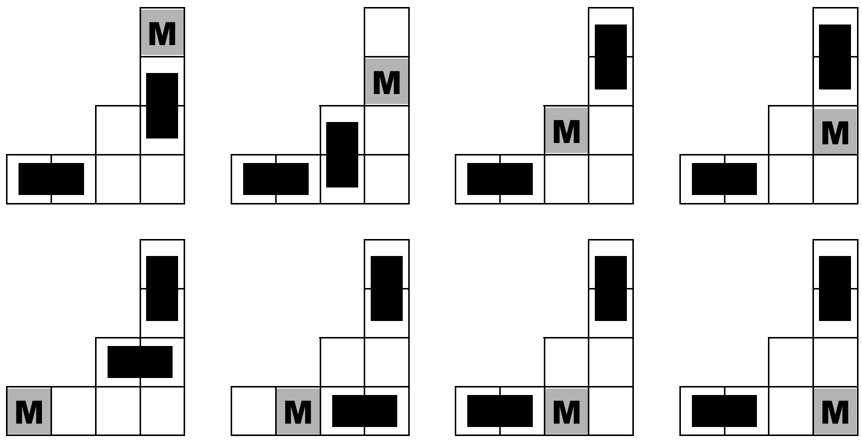


Diagram 3-24
These boards showcase the eight possible moves M can play
(shaded in the above diagrams) from Diagram 3-21c.
Also shown is the alliance’s response to each shaded move.

Diagram 3-24 shows all eight possible moves that M can make from Diagram 3-21c. Also shown is how the alliance should respond to those shaded moves in each board (we will show later in this chapter that the alliance’s response in each diagram is a winning strategy for the alliance).

Note that after 2.25 turns, there are three spaces left in every board with M to play next. The reader can verify that M can play a piece in each of the diagram’s remaining spaces that would give M two reserved spaces and prevent the alliance from placing any more pieces on Slice 2. We established earlier that one of the shaded squares in Diagram 3-20 was a reserved space for M, so the maximum number of reserved spaces for M in Diagram 3-21c is three.

We have now shown that after 2.75 turns, the alliance can hold M to a maximum of three reserved spaces in Slice 2 for L-M-R and R-M-L turn-order games.

This completes the proof of Proposition 3-5.

We have examined L-R-M, R-L-M, M-L-R, M-R-L, L-M-R, and R-M-L turn-order games to where the alliance can no longer move on Slice 2. We have now established that the alliance can hold M to a maximum of three reserved spaces in Slice 2 after three turns, regardless of the turn-order.

Thus, the proofs of Propositions 3-3, 3-4, and 3-5 have proven Proposition 3-2.

3.4 – Endgame Analysis

We will conclude the proof of Proposition 3-1 in this section. In the previous sections, we showed that the alliance can hold M to a maximum of three reserved spaces in Slice 2 after three turns. It is possible that even if the alliance members do not cooperatively block any of M's available moves, they will still have at least one move after M plays on his reserved spaces in Slice 2. However, there are some situations were the alliance must cooperatively block at least one of M's potential moves in order to prevent M from winning. First, let's observe how M's three reserved spaces in Slice 2 affect the endgame of a 4x4x3 Domineering game.

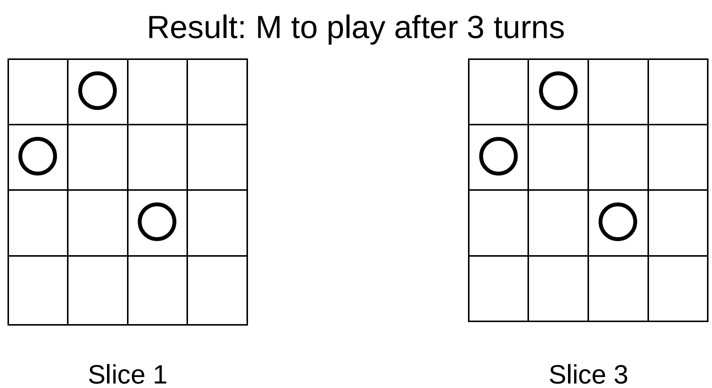


Diagram 3-25
The O's represent potential moves where M can play during his third turn.

Diagram 3-25 shows potential areas in Slices 1 and 3 that M's piece would cover (marked with O's) if he played on his reserved spaces in Slice 2. We are first going to show that if the alliance agrees not to play on any of the potential squares that M can play, each alliance member can place a minimum of five pieces on each slice, as long as one member plays only on one slice and the other only plays on the other slice (for example, if L plays on Slice 1, then R plays on Slice 3).

Lemma: *For any arrangement of three ○'s in corresponding spaces on Slice 1 and Slice 3, we can place at least five L pieces on one slice and at least five R pieces on the other.*

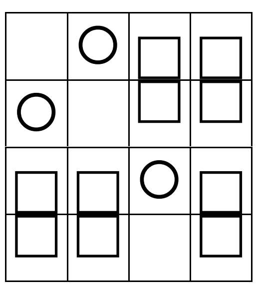
Proof:

Note: Diagrams 3-26a to 3-26c refer to Slice A and Diagrams 3-27a to 3-27c refer to Slice B, where either A = 1 and B = 3, or A = 3 and B = 1.

Case 1: On a slice with ○'s on three distinct columns, we can place two L pieces on the blank column and one L piece on the other columns. That gives us a total of five L pieces. See Diagram 3-26a for an example.

Case 2: On a slice with ○'s that are not on three distinct columns, there are at least two columns that are blank (two L pieces per empty column), and at least one that has less than or equal to one ○ (at least one L piece on this column). This gives us a total of at least five L pieces. See Diagrams 3-26b and 3-26c for examples.

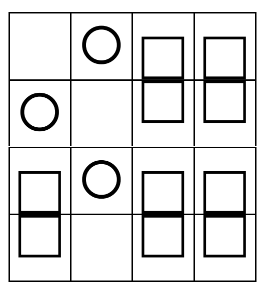
Case 1 example



Potential Slice A

Diagram 3-26a
The ○'s are in three distinct columns. L is able to place five pieces in this configuration

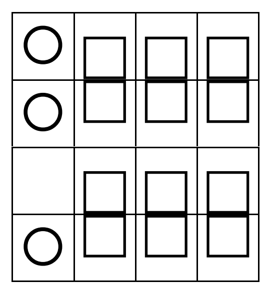
Case 2 example



Potential Slice A

Diagram 3-26b
The ○'s are not in three distinct columns. L is able to place five pieces in this configuration.

Another Case 2 example



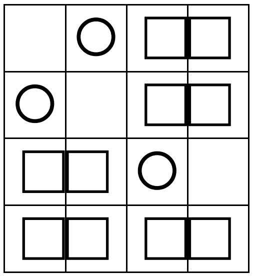
Potential Slice A

Diagram 3-26c
The ○'s are on the same column. L is able to place six pieces in this configuration.

Case 3: On a slice with ○'s on three distinct rows, we can place two R pieces on the blank row and one R piece on the other rows. That gives us a total of five R pieces. See Diagram 3-27a for an example.

Case 4: On a slice with ○'s that are not on three distinct rows, there are at least two rows that are blank (two R pieces per empty row), and at least one that has less than or equal to one ○ (at least one R piece on this row). This also gives us a total of at least five R pieces. See Diagrams 3-27b and 3-27c for examples. ◻

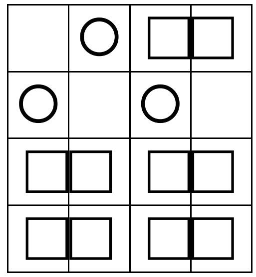
Case 3 example



Potential Slice B

Diagram 3-27a
The ○'s are in three distinct rows. R is able to place five pieces in this configuration

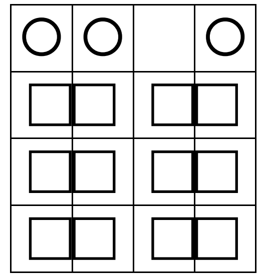
Case 4 example



Potential Slice B

Diagram 3-27b
The ○'s are not in three distinct rows. R is able to place five pieces in this configuration.

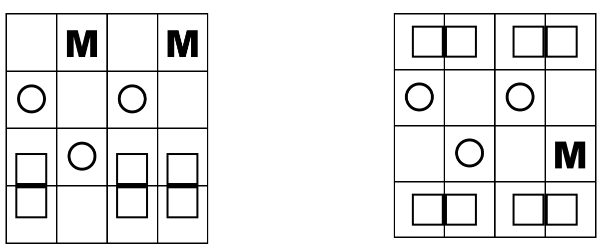
Another Case 4 example



Potential Slice B

Diagram 3-27c
The ○'s are on the same row. R is able to place six pieces in this configuration.

Although we showed that L and R can each place at least five pieces on their respective slices regardless of how M's three potential spaces are configured on the board, we did not take into account the three M pieces that are already on the board. Recall that we proved that the alliance can limit M to a maximum of three reserved spaces on Slice 2 after three turns. After three complete turns, M has moved three times; thus, there are three M pieces on the board. Those three pieces will affect how many pieces each alliance member can play on each slice, as demonstrated in the following diagram.



Slice A Slice B

Diagram 3-28
The three M pieces affect how many pieces each alliance member can play on each slice.

Diagram 3-28 represents a potential 4x4x3 game after three turns. The two M pieces on Slice A reduce L's five potential moves that will not cover a potential M move to three, and the one M piece on Slice B reduces R's five potential moves that will not cover a potential M move to four.

Observation: *Each M piece on a slice reduces the number of an alliance member's potential moves (one that doesn't block ○) on that slice.*

The reader may wonder why the alliance is agreeing to play with "one-hand tied" by not cooperatively covering M's remaining spaces. If we can show that the alliance still has one move left after M's final move even if it does not cover M's remaining moves, then we have definitively proven that the alliance can prevent M from winning. Next, we will show the effectiveness of this "one-hand tied" strategy in M-L-R and M-R-L games.

In Section 3.2, we showed in Diagram 3-17 that after the third turn in an M-L-R and M-R-L game, M has three reserved spaces on Slice 2 and only one of the alliance members was able to move on Slice 2. This forces the other alliance member to play on either Slice 1 or Slice 3 in his third turn. Where that alliance member should play in his third turn depends on how many M pieces are already on Slices 1 and 3. Since there are three M pieces on the board after three turns, there will either be no M pieces on Slice A and three M pieces on Slice B, or there will be one M piece on Slice A and two M pieces on Slice B (where A = 1 and B = 3, or A = 3 and B = 1). The alliance member's strategy is as follows:

Strategy: *The alliance member forced to play on either Slice 1 or 3 should choose the slice that has the least number of M pieces and keep playing on that slice for the rest of the game until a move there is no longer available.*

For example, let's say R cannot place a piece on Slice 2 during his third turn. The board has one M piece on Slice 1 and two M pieces on Slice 2. Thus, R chooses to play on Slice 1 in his third turn, and will keep playing on that slice for the rest of the game. The other alliance member should not play on the same slice where his partner played, because such a move would block two potential moves for his partner.

Strategy: *The other alliance member should play on the slice that has more M pieces for the rest of the game until a move is no longer available.*

In our previous example, R chose to play on Slice 1 because it has fewer M pieces. Therefore, L needs to play on Slice 3 for the rest of the game to avoid overlapping R's potential moves on Slice 1 unless there are no more moves available on Slice 3.

In our analysis of M-L-R and M-R-L games, there are two possible scenarios to examine:

- In games with no M pieces on one slice and three M pieces on the other, the player who cannot play on Slice 2 in his third turn plays on the slice with no M pieces.

- In games with one M piece on one slice and two M pieces on the other, the player who cannot play on Slice 2 in his third turn plays on a slice with one M piece.

We will start with the first case for M-L-R and M-R-L games, where an alliance member is forced to play on a Slice 1 or 3 with no M pieces.

Number of remaining moves in each turn of M-L-R & M-R-L games						
Turn Number	M-L-R Game: Who Plays Next	M-R-L Game: Who Plays Next	Number of Remaining Moves			
			For M	For 2nd Player	For 3rd Player	For Alliance
3	M to play next	M to play next	3	2	4	6
3.5	L to play next	R to play next	2	2	4	6
3.75	R to play next	L to play next	2	1	4	5
4	M to play next	M to play next	2	1	3	4
4.5	L to play next	R to play next	1	1	3	4
4.75	R to play next	L to play next	1	0	3	3
5	M to play next	M to play next	1	0	2	2
5.5	L to play next	R to play next	0	0	2	2

Diagram 3-29

Diagram 3-29 reflects 4x4x3 M-L-R and M-R-L games where the third player is forced to play on either Slice 1 or 3 that has no M pieces during the third turn. The second player will play on the slice that has three M pieces in his next turn, and both alliance members will play on their respective slices (for example, if L plays on Slice 1, then R plays on Slice 3) for the rest of the game unless a move is no longer available on that slice. The third player had five available moves on the slice with no M pieces, but because he was forced to play on that slice during the third turn, his five available moves are reduced to four. The second player will play on the other slice that has three M pieces, so his five available moves are reduced to two. The reader will note in the chart that during the fifth turn, when M plays on his last reserved space, the third player has two moves left in his slice. Since an alliance member has at least one move after M played his final piece, this is a loss for M.

This chart presumes that the third player of M-L-R and M-R-L games was forced to play on Slice 1 or 3 with no M pieces during his third turn. If the second player was forced to play on Slice 1 or 3 with no M pieces during his third turn, the total number of remaining moves for the alliance would remain the same after each turn, as indicated in the following chart.

Number of remaining moves in each turn of M-L-R & M-R-L games						
Turn Number	M-L-R Game: Who Plays Next	M-R-L Game: Who Plays Next	Number of Remaining Moves			
			For M	For 2nd Player	For 3rd Player	For Alliance
3	M to play next	M to play next	3	4	2	6
3.5	L to play next	R to play next	2	4	2	6
3.75	R to play next	L to play next	2	3	2	5
4	M to play next	M to play next	2	3	1	4
4.5	L to play next	R to play next	1	3	1	4
4.75	R to play next	L to play next	1	2	1	3
5	M to play next	M to play next	1	2	0	2
5.5	L to play next	R to play next	0	2	0	2

Diagram 3-30

The chart in Diagram 3-30 shows that the alliance has at least one more move after M's final move, which is a loss for M.

Next, we will analyze the second case for M-L-R and M-R-L games, where an alliance member is forced to play on a Slice 1 or 3 with one M piece.

Number of remaining moves in each turn of M-L-R & M-R-L games						
Turn Number	M-L-R Game: Who Plays Next	M-R-L Game: Who Plays Next	Number of Remaining Moves			
			For M	For 2nd Player	For 3rd Player	For Alliance
3	M to play next	M to play next	3	3	3	6
3.5	L to play next	R to play next	2	3	3	6
3.75	R to play next	L to play next	2	2	3	5
4	M to play next	M to play next	2	2	2	4
4.5	L to play next	R to play next	1	2	2	4
4.75	R to play next	L to play next	1	1	2	3
5	M to play next	M to play next	1	1	1	2
5.5	L to play next	R to play next	0	1	1	2

Diagram 3-31

Diagram 3-31 reflects 4x4x3 M-L-R and M-R-L games where the either the second or third player is forced to play on either Slice 1 or 3 that has one M piece during the third turn. The other alliance member will play on the other slice that has two M pieces in his next turn. Both alliance members will then play on their respective slices (for example, if L plays on Slice 1, then R plays on Slice 3) for the rest of the game unless a move is no longer available on that slice. The alliance member forced to play on the slice with one M piece had four available moves, but because he was forced to play on that slice during the third turn, his four available moves were reduced to three. The other alliance member will play on the other slice that has two M pieces, so his five available moves are reduced to three. The reader will note that during the fifth turn, when M plays on his last reserved space, both alliance members have one move left. Since the alliance has at least one move after M played his final piece, this is a loss for M.

We have shown that the alliance can prevent M from winning all possible M-L-R and M-R-L games, even if they play "one-hand tied" and do not cooperatively cover any of M's three remaining moves. We will next discuss how the alliance can prevent M from winning L-M-R and R-M-L games.

Unfortunately, we cannot apply the "one-hand tied" strategy for L-M-R and R-M-L games to show that the alliance can prevent M from winning. In Section 3.3, we showed in Diagrams 3-22, 3-23, and 3-24 that after 2.75 turns, M will place his third piece on

Slice 2, which prevents both L and R to play any more pieces on Slice 2. If the alliance members were to ignore the opportunity to cooperatively block M in their next turn, neither will be able to play a piece after M plays his last piece, which results in a win for M.

In our analysis of L-M-R and R-M-L games, there are two possible scenarios to examine:

- In games with no M pieces on one slice and three M pieces on the other, the third player plays on the slice with no M pieces in his third turn.

- In games with one M piece on one slice and two M pieces on the other, the third player plays on the slice with one M piece in his third turn.

We will start with the first case for L-M-R and R-M-L games, where the third player is forced to play on a Slice 1 or 3 with no M pieces.

Number of available moves for each turn in L-M-R & R-M-L game						
Turn Number	L-M-R Game: Who Plays Next	R-M-L Game: Who Plays Next	Number of Remaining Moves			
			For M	For 1st Player	For 3rd Player	For Alliance
2.75	R to play next	L to play next	3	2	5	7
3	L to play next	R to play next	3	2	4	6
3.25	M to play next	M to play next	3	1	4	5
3.75	R to play next	L to play next	2	1	4	5
4	L to play next	R to play next	2	1	3	4
4.25	M to play next	M to play next	2	0	3	3
4.75	R to play next	L to play next	1	0	3	2
5	L to play next	R to play next	1	0	2	2
5.25	M to play next	M to play next	1	0	0	0

Diagram 3-32

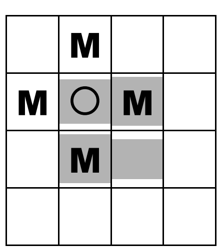
Diagram 3-32 presumes that there are three M pieces on one slice and the third player decided to play on a slice that has no M pieces. The third player has five potential moves, as his slice has no M pieces, and the first player has two potential moves, as his slice has three M pieces. The problem for the alliance occurs after 4.75 turns. After the third player plays on his slice, there are two potential moves left. However, the first player has no moves left on his respective slice, so he is forced to play on the other slice, which will overlap the third player's remaining two moves. Since neither alliance member has any moves left after M plays his final piece, this game is a win for M. Therefore, the alliance must cooperatively block at least one of M's moves to prevent M from winning in L-M-R and R-M-L games.

A reserved move for M is one that neither L nor R can block at any point during the game. M can *protect* a potential move (and thus reserve it) if he places pieces around the potential move in such a way that neither alliance member can overlap it. We are going to prove the following to show that the alliance can block at least one of M's potential moves:

Proposition 3-6: *M cannot protect all of his three potential moves with his three M pieces on a 4x4 slice.*

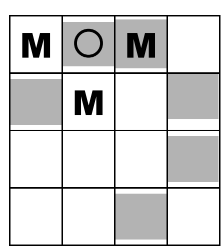
Proof:

We previously showed that, after 2.75 turns in L-M-R and R-M-L games, M will have three reserved spaces in Slice 2 and three M pieces on the board. There exists no configuration on a 4x4 slice where M's three pieces will protect all three of his potential moves. This is because in order for M to protect one of his moves on either Slice 1 or Slice 3, he needs at least two pieces.



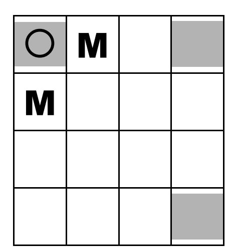
Potential Slice 1 or 3

Diagram 3-33a
O's in center spaces need four M pieces to be protected



Potential Slice 1 or 3

Diagram 3-33b
O's in spaces that are neither corner nor center need three M pieces to be protected



Potential Slice 1 or 3

Diagram 3-33c
O's in corner spaces need only two M pieces to be protected

Diagrams 3-33a to 3-33c show how many M pieces are needed to protect a potential move depending on its location. Diagram 3-33a is not possible because we stated that M only has three pieces on the board. Diagram 3-33b has all of M's three pieces protecting only one reserved space, but this leaves the other potential moves unprotected and can be covered by an alliance member. Diagram 3-33c is the best of the three diagrams because M protected a potential move with just two M pieces. However, if we were to place another ○ on Diagram 3-33c, M cannot protect it with the third M piece because we showed using Diagrams 3-33a to 3-33c that M needs at least two pieces to protect a reserved space. If M cannot protect two potential moves with his three pieces, he certainly cannot protect three potential moves with his three pieces. Therefore, we can confidently state that the alliance can cooperatively block one of M's potential moves in its next turn. ◻

This concludes the proof of Proposition 3-6.

Let's reexamine the first case for L-M-R and R-M-L games, where the third player is forced to play on a Slice 1 or 3 with no M pieces, and have the alliance cooperatively block one of M's available moves.

Number of available moves for each turn in L-M-R & R-M-L game						
Turn Number	L-M-R Game: Who Plays Next	R-M-L Game: Who Plays Next	Number of Remaining Moves			
			For M	For 1st Player	For 3rd Player	For Alliance
2.75	R to play next	L to play next	3	2	5	7
3	L to play next	R to play next	3	2	4	6
3.25	M to play next	M to play next	2	1	4	5
3.75	R to play next	L to play next	1	1	4	5
4	L to play next	R to play next	1	1	3	4
4.25	M to play next	M to play next	1	0	3	3
4.75	R to play next	L to play next	0	0	3	3

Diagram 3-34

Diagram 3-34 presumes that there are three M pieces on one slice and the third player plays on the slice with no M pieces. We will assume that the alliance's attempt to cooperatively block one of M's moves will reduce each member's available moves on each slice by one. After 2.75 turns, the third player has five potential moves, as his slice has no M pieces, and the first player has two potential moves, as his slice has three M pieces. M will have two available moves because the alliance will cooperatively block one of his three remaining moves in Turns 2.75 and 3. One can see that after 4.75 turns, the alliance has at least one more move after M plays his final piece, resulting in a loss for M.

Next, we will analyze the second case for L-M-R and R-M-L games, where the third player is forced to play on a Slice 1 or 3 with one M piece.

Number of available moves for each turn in L-M-R & R-M-L game						
Turn Number	L-M-R Game: Who Plays Next	R-M-L Game: Who Plays Next	Number of Remaining Moves			
			For M	For 1st Player	For 3rd Player	For Alliance
2.75	R to play next	L to play next	3	3	4	7
3	L to play next	R to play next	3	3	3	6
3.25	M to play next	M to play next	2	2	3	5
3.75	R to play next	L to play next	1	2	3	5
4	L to play next	R to play next	1	2	2	4
4.25	M to play next	M to play next	1	1	2	3
4.75	R to play next	L to play next	0	1	2	3

Diagram 3-35

Diagram 3-35 presumes that there are two M pieces on one slice and the third player plays on the slice with one M piece. After 2.75 turns, the third player has four potential moves, as his slice has one M piece, and the first player has three potential moves, as his slice has two M pieces. M will have two available moves because the alliance will cooperatively block one of his three remaining moves in Turns 2.75 and 3. The total number of alliance moves for each turn in this chart is the same as those listed on Diagram 3-34. The alliance has at least one more move after M plays his last piece, which results in a loss for M.

We have demonstrated that the alliance can prevent M from winning all possible L-M-R and R-M-L games as long as they cooperatively block at least one of M's three available moves. We will next discuss how the alliance can prevent M from winning L-R-M and R-L-M games.

For L-R-M and R-L-M games, we demonstrated in Section 3.2 that the alliance can limit M to two reserved spaces on Slice 2 after 2.5 turns. This means that after M plays on his third turn, he will only have one move left. The alliance has a least two potential moves in each slice, since one slice could have up to three M pieces, which would reduce an alliance member's five potential moves to two. Thus, even if the alliance members decide not to cooperatively block M's last remaining move, they are guaranteed to move after M plays his final piece, which results in a loss for M.

Therefore, we have shown that for all possible games on a 4x4x3 board, where M plays on the 3 component, M cannot force a win if the alliance teams up against him and uses the strategies we described in this chapter.

This completes the proof of Proposition 3-1.

4.0 – Strategies to Solve a 4x4x3 Domineering Game: Part 2

In the previous chapter, we proved that the alliance can prevent M from winning a 4x4x3 game where M plays on the 3 component (see Chapter 3 for an explanation of what it means to play on a component). To complete the proof that a 4x4x3 game is solved to be a $\mathcal{Q}$ game, we are going to prove the following in this chapter:

Proposition 4-1: *M is unable to force a win in a 4x3x4 game, where M plays on the 4 component, if the L & R alliance teams up against him.*

4.1 – Part 1 of Alliance's Winning Strategy

The first part of L & R's winning strategy against M is similar to the first part of the alliance's winning strategy against M in 3x3x3 and 4x4x3 (where M plays on the 3 component) games, which is to cover as many spaces of Slice 2 as possible. However, because M is able to move on four slices instead of three, the alliance must now cover two slices of the game board to block M from making any more moves. We will show later in this chapter which two slices are most effective to stopping M.

The second part of the alliance's winning strategy is also similar to the second part of the alliance's winning strategy we covered in previous chapters, which is that the alliance should cover the corresponding squares above and below any remaining M playable spaces to block M from playing on those squares. However, there are certain situations in the winning strategy where the alliance is only able to block above but not below or below but not above remaining M playable moves.

Throughout this chapter, we will use the phrase "L & R should play…" in order to describe the strategy L & R use to prevent M from winning,

To assist in the analysis of a 4x3x4 game (where M plays on the 4 component), we created an application that simulates three players playing a 4x3x4 Domineering game. This application was developed because this type of analysis contains too many game tree branches to review by hand. The application considers all possible moves M can make during a game and uses a heuristic to determine moves L and R can make in response to each of M's possible moves. The goal of the heuristic is to pick moves for the alliance that will lead to its victory.

This heuristic differs from the winning strategies devised in the previous chapters such that not every move in the heuristic is necessarily a best-play move for the alliance. This is because the heuristic was designed to be generic enough to avoid writing special-case responses to a multitude of specific scenarios. The reader should be aware that even if a move made by the heuristic does not appear to be a best-play move for the alliance, it is still designed to be a winning move for the alliance.

The first part of the alliance's winning strategy is as follows:

Strategy: *Cover as much of Slices 1 & 3 as possible for the duration of the game until L and R are unable to move on either slice.*

It is crucial that the alliance commits to playing on either Slices 1 or 3 until they cannot move on those slices anymore. Deciding which of the two slices to play will depend on where M last played.

M will play on Slices j and $j+1$ in his turns, where j = 1, 2, or 3. If j = 1 or 3, the alliance should play on the same Slice j. If j = 2, then the alliance should play on Slice 3. However, if an alliance member is unable to play on one of the slices in the Slices 1 & 3 pair, then he should move on the other slice in the pair if a move is available. For example, let's say M played on Slices 1 and 2. According to our strategy, the alliance should move on Slice 1 on its next turn. However, if L or R is unable to move on Slice 1, then that alliance member should play on Slice 3 if a move is available. If an alliance member is unable to play on either Slice 1 or Slice 3, then he must follow the second part of the alliance's winning strategy, which we will cover in the next section.

In the following pages, we will explain the coordinate system of our 3D Domineering simulation application so that we can accurately describe where L and R should play on Slices 1 and 3.

In the application, a move is represented by a three-dimensional coordinate, (x, y, z), where x represents the column of a slice, y represents the row of a slice, and z represents the actual slice. For example, a coordinate that is labeled (0, 0, 0) represents the space located at the first column and first row of the first slice in a 3D Domineering game. The coordinate system in the application follows a zero-based index, which is why the first column, first row, first slice of a 3D Domineering game is (0, 0, 0) and not (1, 1, 1). The following diagram demonstrates a 3x3x3 Domineering game with each space labeled using our 3D Domineering coordinate system.

A 3x3x3 Domineering Game

0,0,0	1,0,0	2,0,0
0,1,0	1,1,0	2,1,0
0,2,0	1,2,0	2,2,0

Slice 1

0,0,1	1,0,1	2,0,1
0,1,1	1,1,1	2,1,1
0,2,1	1,2,1	2,2,1

Slice 2

0,0,2	1,0,2	2,0,2
0,1,2	1,1,2	2,1,2
0,2,2	1,2,2	2,2,2

Slice 3

Diagram 4-1

Each square is labeled with the 3D Domineering coordinate system used by the simulation application. The shaded squares indicate a possible move made by L, R, and M.

You may have noticed that while a move is represented by one three-dimensional coordinate, a piece in a 3D Domineering game takes two spaces. The application is able to identify the second half of a move based on the player who made the move. For example, if L makes a move at (0, 0, 0), the application knows that the second-half of that move is on (0, 1, 0). If R makes a move at (0, 1, 1), the application knows that the second-half is on (1, 1, 1). And if M makes a move in (2, 2, 1), the application knows that the second-half is on (2, 2, 2) (see shaded squares in Diagram 4-2). Keep this in mind as we describe later in this chapter what moves the alliance members should make for their winning strategy.

Now that we explained how a move is described using our application's 3D Domineering coordinate system, let us examine where L and R should play on a playable slice (either Slice 1 or Slice 3) using the following chart.

Alliance’s Winning Strategy Details and How They Depend on the Number of Pieces in a Given Slice					
Number of pieces on alliance's next playable slice (Slice 1 or 3)			Special condition	The moves L & R should make as part of the alliance’s winning strategy.	
# of L pieces	# of R pieces	# of M pieces		L’s move	R’s move
0	0	0	None	(0, 1, 1 or 3)	(0, 0, 1 or 3)
0	0	1	M piece is on Column 3 or 4	(0, 1, 1 or 3)	(0, 0, 1 or 3)
0	0	1	M piece is on Column 1 or 2	(3, 0, 1 or 3)	(2, 2, 1 or 3)
1	0	0	None	(3, 0, 1 or 3)	(2, 2, 1 or 3)
0	1	0	None	(3, 0, 1 or 3)	(2, 2, 1 or 3)
1	1	0	None	(2, 0, 1 or 3)	(3, 1, 1 or 3)
1	0	1	M piece is on Column 3 or 4	(1, 1, 1 or 3)	(0, 0, 1 or 3)
1	0	1	M piece is on Column 1 or 2	(3, 0, 1 or 3)	(2, 2, 1 or 3)
0	1	1	M piece is on Column 3 or 4, Row 1	(0, 1, 1 or 3)	(2, 2, 1 or 3)
0	1	1	M piece is on Column 3 or 4, but not Row 1	(0, 1, 1 or 3)	(2, 0, 1 or 3)
0	1	1	M piece is on Column 1 or 2	(3, 0, 1 or 3)	(2, 2, 1 or 3)
1	1	1	See Diagrams 4-10 and 4-11		
2	1	0	None	(2, 0, 1 or 3)	(0, 0, 1 or 3)
1	2	0	None	(2, 0, 1 or 3)	(0, 1, 1 or 3)
2	2	0	None	First player of alliance moves on slice. Other must play on a different slice.	
1	1	2	None	First player of alliance plays move on slice that won’t block second player from moving on same slice.	
Total >= 5			None	If both players cannot move on same slice, then first player who can move on slice should take it while other player plays on different slice. If no move is available, then both members play on a different slice.	

Diagram 4-2

This chart summarizes the moves the alliance should make on Slice 1 or 3 as part of its winning strategy.

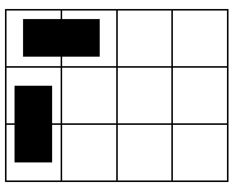
The chart in Diagram 4-2 summarizes the first part of the alliance’s winning strategy. It shows which moves the alliance should make when they can play on Slice 1 or 3. Those moves are dependent on how many pieces are already on that slice and, in some cases, where those pieces are located. We will describe the alliance’s moves demonstrated in this chart with more detail in the following pages, starting with the alliance playing on an empty slice.

1) Alliance plays on an empty slice

There are two situations when the alliance members will play on an empty slice. The first situation is at the start of an L-R-M, R-L-M, L-M-R or R-M-L game. The following demonstrates where the alliance should play at the start of those games.

First moves L and R should make - M to play next

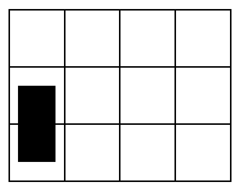
L-R-M or R-L-M Game



Slice 1

Diagram 4-3a
The first moves L and R should make in this game.

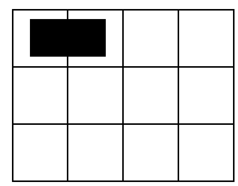
L-M-R Game



Slice 1

Diagram 4-3b
The first move L should make in this game.

R-M-L Game



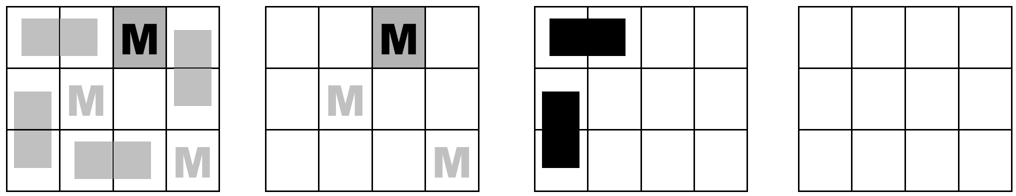
Slice 1

Diagram 4-3c
The first move R should make in this game.

The alliance's winning strategy relies on where M last played, but since M has not moved yet in this particular example, we picked Slice 1 as the place where the alliance will first move. Note that R moved on the upper-left corner of the slice and L moved on the first column below R's piece in their turns. These moves (indicated in Diagrams 4-3a, 4-3b, and 4-3c) are where the alliance members should play if they encounter an empty slice.

The next example demonstrates the second situation where the alliance will play on an empty slice. There are certain 4x3x4 games where M moves on one of the slices of the Slice 1 & 3 pair that neither alliance member can move on in their next turn, and the other slice in the pair happens to be empty. The following diagram demonstrates a potential 4x3x4 game where such a scenario can occur.

A sample 4x3x4 game (M-L-R/M-R-L)
M to play after three turns



Slice 1 Slice 2 Slice 3 Slice 4

Diagram 4-4
On M's third turn, M plays on Slice 1.
L and R can't move on Slice 1, so they move on Slice 3, which is empty.

In Diagram 4-4, M has made three moves on Slices 1 and 2. L and R have been able to move on Slice 1 in their first two turns, but are unable to play on that slice again in their third turn. Therefore, according to our winning strategy, the alliance must play on Slice 3, which M has not played in this sample game. Note that the moves the alliance played on Slice 3 are the ones we indicated in Diagram 4-3a.

2) Alliance plays on a slice with one piece

If the alliance were to play on Slice 1 or 3 that already has one M piece, then the following demonstrates the winning strategy for the alliance.

M to play next

Diagram 4-5a
Either Slice 1 or 3

Diagram 4-5b
Either Slice 1 or 3

If M plays on any of the shaded spaces of an empty slice
(Slice 1 or 3), the alliance should respond accordingly.

If M played on a previously empty slice (either Slice 1 or 3), the alliance's response will depend on what column M's piece is on. If M plays on the third or fourth row of the slice, then R should play on the upper-left corner of the slice and L should play a piece below it on the first column (as shown in Diagram 4-5a). If M plays on the first or second column of the slice, then R should play on the lower-right corner of the slice and L should play a piece above it on the fourth column (as shown in Diagram 4-5b).

The reader will notice that L and R play their pieces away from where M last played his piece in Diagrams 4-5a and 4-5b. The primary reason is that it permits the alliance to place their pieces along the edge of a slice. This prevents M from trying to reserve spaces on the corners of the slice.

If the alliance were to play on Slice 1 or 3 that already has one L piece or one R piece, then the following demonstrates the winning strategy for the alliance.

M to play next

Diagram 4-6a
Either Slice 1 or 3

Diagram 4-6b
Either Slice 1 or 3

The above diagrams show where L and R should play on a Slice 1 or 3 if it already has an L or R piece.

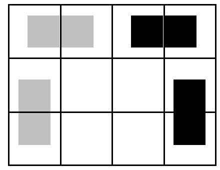
In both diagrams, R plays on the lower right corner of the slice and L plays above it on the fourth column. We established in Diagrams 4-3b and 4-3c where L and R would play on an empty slice, so the gray pieces in Diagrams 4-6a and 4-6b reflect those moves.

3) Alliance plays on a slice with two pieces

We will now describe the alliance's winning strategy if their next playable slice already has two pieces.

If the alliance were to play on Slice 1 or 3 that already has an L piece and an R piece, then the following demonstrates the winning strategy for the alliance.

M to play next



Either Slice 1 or 3

Diagram 4-7

L and R should play on the following spaces in this scenario

We indicated in Diagram 4-1a where L and R should play if they start on an empty slice, so the grayed L and R pieces in the above diagram reflect those moves. The alliance's next move if they were forced to play in this Slice 1 or 3 is for R to play on the upper-right corner and L to play below it on the fourth column (see Diagram 4-7).

The following diagram demonstrates the alliance's winning strategy if the alliance were to play on Slice 1 or 3 that already has an L piece and an M piece.

M to play next

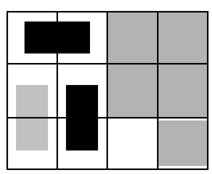


Diagram 4-8a

Either Slice 1 or 3

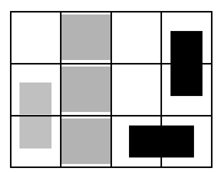


Diagram 4-8b

Either Slice 1 or 3

If M plays on any of the shaded spaces, the alliance should respond accordingly.

We indicated in Diagram 4-3b where L should play if he starts on an empty slice, so the grayed L pieces in both Diagrams 4-8a and 4-8b reflect that move. In Diagram 4-8a, the shaded squares represent M playing on either the third or fourth column, so the alliance's winning strategy is for R to play on the upper-left corner, and L to play below that piece on the second column. In Diagram 4-8b, the shaded squares represent M playing on either the first or second column, so the alliance's winning strategy is for R to play on the lower-right corner and L to play above it on the fourth column.

The following diagram demonstrates the alliance's winning strategy if the alliance were to play on a Slice 1 or 3 that already has an R piece and an M piece.

M to move next

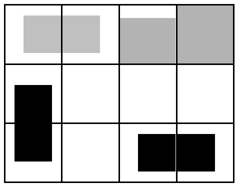


Diagram 4-9a
Either Slice 1 or 3

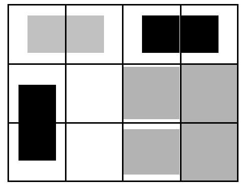


Diagram 4-9b
Either Slice 1 or 3

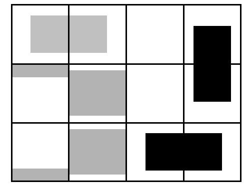


Diagram 4-9c
Either Slice 1 or 3

If M plays on any of the shaded spaces, the alliance should respond accordingly.

We indicated in Diagram 4-3c where R should play if he starts on an empty slice, so the grayed R pieces in Diagrams 4-9a, 4-9b, and 4-9c reflect that move. In Diagram 4-9a, the shaded squares represent M playing on the first row (either third or fourth column), so the alliance's winning strategy is to respond with R playing on the lower-right corner, and L playing on the first column. In Diagram 4-9b, the shaded squares represent M playing on either the third or fourth column that isn't on the first row, so the alliance's winning strategy is to respond with R playing on the upper-right corner and L paying on the first column. In Diagram 4-9c, the shaded squares represent M playing on the first or second column, so the alliance's winning strategy is to respond with R playing on the lower-right corner and L playing above it on the fourth column.

4) Alliance plays on a slice with three pieces

We will now describe the alliance's winning strategy if their next playable slice already has three pieces.

Diagrams 4-10 and 4-11 demonstrate the alliance's winning strategy if the alliance were to play on Slice 1 or 3 that already has an L piece, an R piece, and an M piece.

All boards represent a potential Slice 1 or 3
M to play next on each one

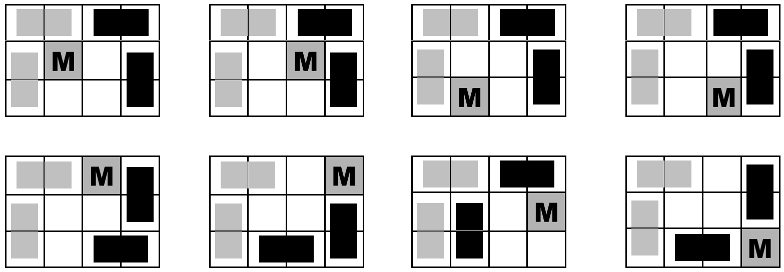


Diagram 4-10
These boards show how the alliance should respond to the eight possible moves by M (indicated by shaded square) in this scenario.

In Diagram 4-10, each board has a grayed R piece on the upper-left corner and a grayed L piece below it on the first column. We indicated in Diagrams 4-3a and 4-5a the scenarios where the alliance would make those moves. Each board in Diagram 4-10 shows where the alliance can play in their next turn depending on the location of M's piece.

All boards represent a potential Slice 1 or 3
M to play next on each one

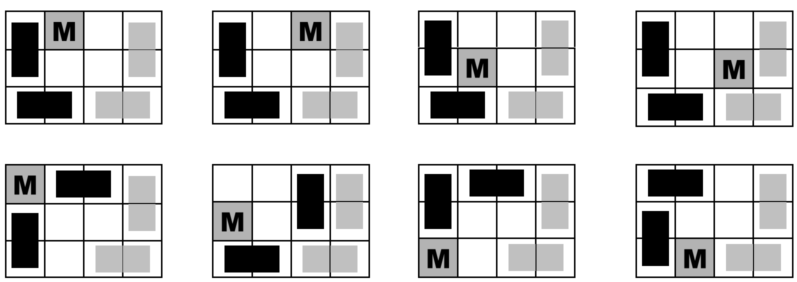


Diagram 4-11
These boards show how the alliance should respond to the eight possible moves by M (indicated by shaded square) in this scenario.

In Diagram 4-11, each board has a grayed R piece on the lower-right corner and a grayed L piece above it on the fourth column. We indicated in Diagram 4-5b the scenario where the alliance would make those moves. Each board in Diagram 4-11 shows how the alliance can play in their next turn depending on the location of M's piece.

The following diagram demonstrates the alliance's winning strategy if the alliance were to play on Slice 1 or 3 that already has two L pieces and one R piece.

M to play next

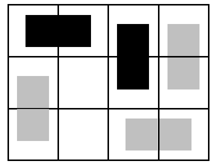


Diagram 4-12
Either Slice 1 or 3

The above diagram shows where L and R should play,
if the above slice already has the grayed piece configuration

The three grayed pieces in Diagram 4-12 were moves that were already indicated in Diagram 4-6a. The alliance's winning strategy is for R to play on the upper-left corner and L to play on the third column.

The following diagram demonstrates the alliance's winning strategy if the alliance were to play on a Slice 1 or 3 that already has two R pieces and one L piece.

M to play next

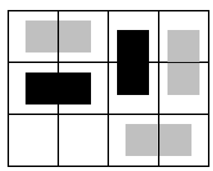


Diagram 4-13
Either Slice 1 or 3

The above diagram shows where L and R should play,
if the above slice already has the grayed piece configuration

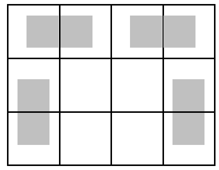
The three grayed pieces in Diagram 4-13 were moves that were already indicated in Diagram 4-6b. The alliance's winning strategy is for R to play on the second row across the first and second columns and L to play on the third column.

5) Alliance plays on a slice with four pieces

We will now examine the alliance's winning strategy if their next playable slice already has four pieces.

The following diagram demonstrates the alliance's winning strategy if the alliance were to play on a Slice 1 or 3 that already has two L pieces and two R pieces.

L or R to play next



Either Slice 1 or 3

Diagram 4-14

In this scenario, only one of the alliance members can move in the available spaces.

The gray pieces in Diagram 4-14 were moves that were already indicated in Diagram 4-7. Only one of the alliance members is able to move in this slice. The other member will either have to move on the other slice of the Slice 1 & 3 pair if such a move is available, or move on Slice 2 or Slice 4. If an alliance member is forced to play on either Slice 2 or Slice 4, then that member must follow the second part of the alliance's winning strategy. That strategy will be explained later in this chapter.

If the alliance's next playable slice has four pieces and there is at least one M piece on that slice, then the alliance's winning strategy will be for one alliance member to perform his next available move that will not prevent the next alliance member from playing on the same slice. The following diagram shows an example of this strategy.

Sample 4x3x4 games

M to move next

M-L-R Game

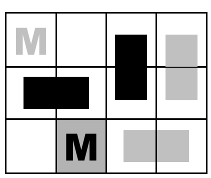


Diagram 4-15a

Either Slice 1 or 3

M-R-L Game

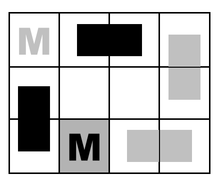


Diagram 4-15b

Either Slice 1 or 3

After M's turn (indicated by shaded square), an alliance member played the next available move that would not block the other alliance member from playing on the same slice.

Diagrams 4-15a and 4-15b show how the alliance members have played pieces that did not prevent the other member from playing on the same slice. The reader will notice that the alliance played different pieces in the two diagrams despite the board configuration being the same before the alliance moved. These moves are not hard-coded—they are made based on what moves are available for each player. To understand why these different moves were made, we need to explain how the application handles available moves for each player.

Before the application simulates a 3D Domineering game, it generates all the possible moves that each player can make. These possible moves are stored in a sorted set that are sorted by the z coordinate first, then the y coordinate, and finally the x coordinate. Therefore, (1, 0, 0) is sorted before (0, 1, 0) in the set, and (1, 2, 0) is sorted before (0, 0, 1). Once the simulation starts, every move made by each player removes available spaces, and those spaces are removed from each player's available moves set if they exist in the set.

Diagram 4-15a shows a sample M-L-R game where L moves next after M plays on the second column, third row of the potential Slice 1 or 3 (indicated by the shaded square). Space (1, 0, 1 or 3) is the first available move in L's sorted set of available moves that L can play on the potential Slice 1 or 3. However, if L were to play on that space, R would not be able to play on that same slice. Since that goes against the alliance’s winning strategy, we will need to find another available for L that would still allow R to play on the same slice. L's next available move on the potential Slice 1 or 3 in his available moves set is (2, 0, 1 or 3), which is an acceptable move according to the alliance’s winning strategy because it allows R to move on the same slice at (0, 1, 1 or 3).

Diagram 4-15b shows a sample M-R-L game where R moves next after M plays on the second column, third row of the potential Slice 1 or 3 (indicated by the shaded square). Space (1, 0, 1 or 3) is the first available move in R’s sorted set of available moves that R can play on the potential Slice 1 or 3. This is an acceptable move according to the alliance’s winning strategy because it allows L to move on that same slice at (0, 1, 1 or 3).

Although the alliance performed different moves when the turn order of the game differed, they are still winning moves because they accomplish the same goal—to cover as many spaces on Slices 1 and 3 as possible to remove playable spaces for M.

6) Alliance plays on a slice with five pieces

We will now examine the alliance's winning strategy if their next playable slice already has five pieces.

The following diagram demonstrates the alliance's winning strategy if the alliance were to play on a Slice 1 or 3 that already has two L pieces, two R pieces, and one M piece.

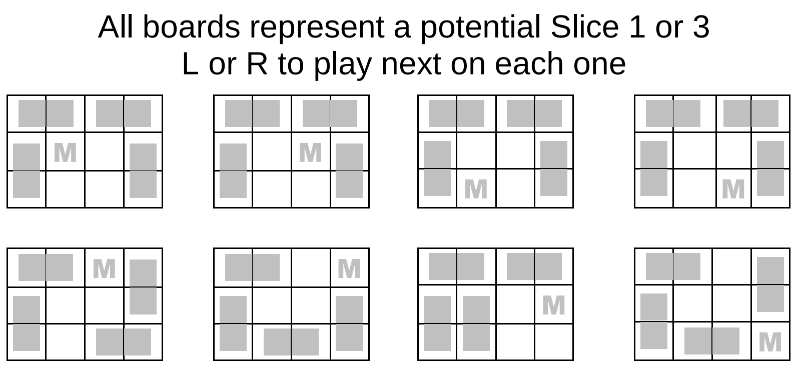


Diagram 4-16
In these scenarios, only one of the alliance members can move in the available spaces.

Diagram 4-16 contain possible board configurations of a potential Slice 1 or 3 with two L pieces, two R pieces, and one M piece that were already indicated by moves made in Diagram 4-10. We excluded showing the other possible board configurations of Diagram 4-11 because they are symmetrical to the board configurations of Diagram 4-10. In each potential Slice 1 or 3 of Diagram 4-16, one can see that only one of the alliance members is able to move in his next turn. The other member will either have to move in the other slice of the Slices 1 & 3 pair (if move is available) or move on Slice 2 or Slice 4. If an alliance member is forced to play on Slice 2 or Slice 4, then that member must follow the second part of the alliance's winning strategy. That second part will be explained later in this chapter.

The following explains the winning strategy for the alliance if the alliance were to play on a potential Slice 1 or 3 that already has five pieces, with two of those pieces belonging to M.

The first thing the alliance will need to check is to see if both players can still move on the same slice. These situations typically occur if M does not attempt to reserve spaces in Slices 1 or 3, as indicated by the sample 4x3x4 game shown below.

A sample 4x3x4 game (L-M-R)
R to play after 3.75 turns

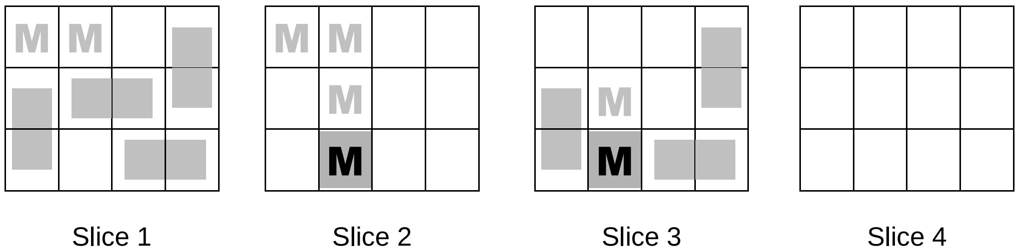


Diagram 4-17
After M's fourth turn (indicated by shaded square), both alliance members can move on Slice 3.

In Diagram 4-17, one can see that both R and L can move on Slice 3 in their next turns.

However, if both alliance members are unable to move on the same potential Slice 1 or 3, then the first alliance member to move next on the slice takes the move, while the other alliance member must move on a different slice. The following shows a sample game where such a scenario can occur.

A sample 4x3x4 game (L-M-R)
R to play after 3.75 turns

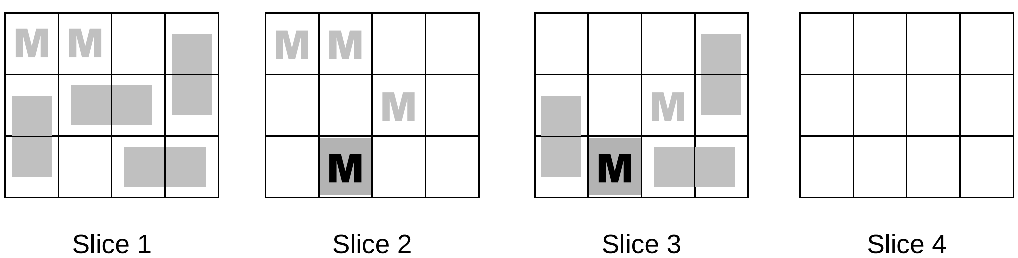


Diagram 4-18
After M's fourth turn (indicated by shaded square), only one alliance member can move on Slice 3.

One can see that after R moves on Slice 3, L is no longer able to move on Slice 3. In this particular example, L cannot move on Slice 1 either, so L must perform the second part of the alliance's winning strategy, which we will cover later.

7) Alliance plays on a slice with six pieces

We will now examine the alliance's winning strategy if their next playable slice already has six pieces.

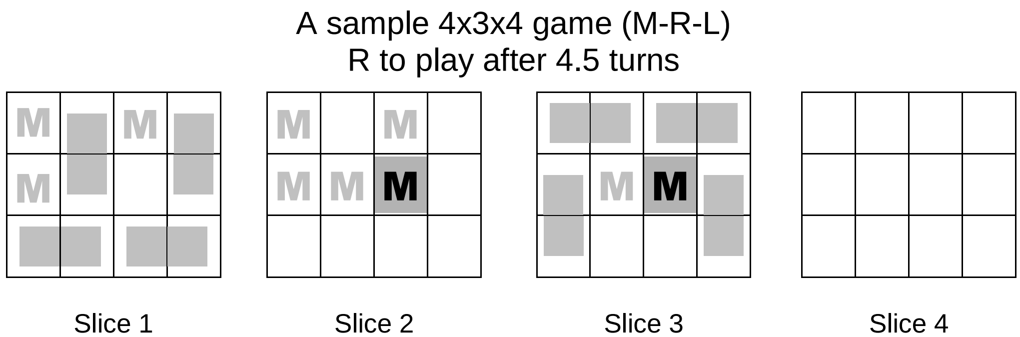


Diagram 4-19
After M's fifth turn (indicated by shaded square), only R can move on Slice 3.

Typically if an alliance member is able to move on a Slice 1 or 3 that has six pieces, then M did not attempt to reserve spaces in that slice, which puts M in a severe disadvantage against the alliance. In those instances, only one of the alliance members is able to move on that slice because the six pieces (usually two L, two R, and two M pieces) occupy ten spaces on a slice, leaving only two remaining spaces left. In Diagram 4-19, only R is able to move on Slice 3, leaving L to play on another slice. In this particular example, however, M is eliminated after R plays on Slice 3 as M has no more remaining moves to play in this game.

We have demonstrated all possible scenarios of the alliance playing on a potential Slice 1 or 3. At some point during the game, the alliance member will not be able to play on either Slice 1 or Slice 3 anymore. Then, the alliance member must follow the second part of the alliance's winning strategy, which we will cover in the next section.

4.2 – Part 2 of Alliance's Winning Strategy

Earlier in this chapter, we mentioned that when an alliance member is no longer able to move on either Slice 1 or 3, he must perform the second part of alliance's winning strategy against M. We continue the proof of Proposition 4-1 by explaining what that second part is.

The second part of the alliance's winning strategy is similar to the second part of the alliance's winning strategy we covered in previous chapters:

Strategy: *The alliance must cover the corresponding squares above and below any remaining moves for M to block him from playing them. However, if the alliance is unable to cooperatively block an available move for M, then both players should strive to either block available M moves that only need one piece to block or partially block an available M move.*

There are two cases to consider when the alliance starts the second part of their winning strategy. The first case is when both alliance members are unable to move on Slices 1 and 3 in their next turn. The second case is when only one alliance member is unable to move on Slices 1 and 3 in his next turn. We will start the analysis of the winning strategy with the first case.

Case 1: Both alliance members can no longer play on Slices 1 and 3

If both alliance members can no longer move on Slices 1 and 3 in their next turn, then they should cooperatively block an available M space in their next turn if the opportunity exists. The following diagram shows an example of this strategy.

A sample 4x3x4 M-L-R game
M to play next after 5 turns

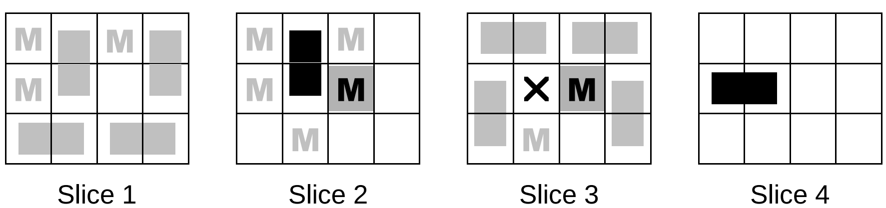


Diagram 4-20
After M's fifth turn (indicated by the shaded square), the alliance plays moves above and below an available M move to block it (indicated by the x)

Diagram 4-20 shows a sample 4x3x4 M-L-R game where the alliance is no longer able to play on Slices 1 and 3 after M's fifth turn (indicated by the shaded square). At this point, the alliance is forced to play on Slices 2 and 4, so their winning strategy is to cooperatively block one of M's remaining moves by playing pieces above and below an available space. In Diagram 4-20, L and R have chosen the first moves from their respective ordered available moves sets that will block one of the two remaining M moves on Slice 3 (indicated by the ‘x’).

However, simply choosing the first available L and R moves that cooperatively block M is not an optimal strategy for the alliance. The following diagrams will demonstrate why.

A sample 4x3x4 game (L-R-M)
M to play next after 4.5 turns

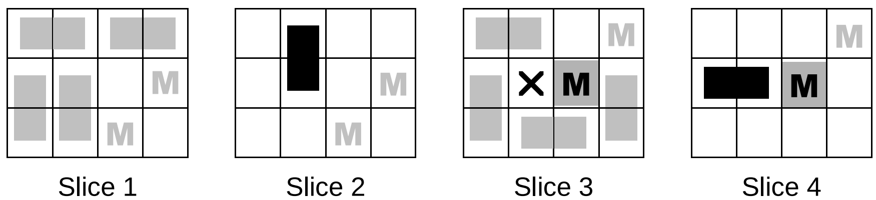


Diagram 4-21
After M’s fourth turn (indicated by the shaded square), the alliance plays moves above and below an available M move in Slice 3 (as indicated by the x).

In Diagram 4-21, the alliance cooperatively blocks one of M’s remaining moves on Slice 3 after M’s fourth turn. These moves were selected by picking the first moves from L and R’s ordered available moves set that can block one of M’s available moves. Those moves are not an optimal winning strategy for the alliance because there is another pair of moves that the alliance can make that can block two of M’s available moves.

A sample 4x3x4 game (L-R-M)
M to play next after 4.5 turns

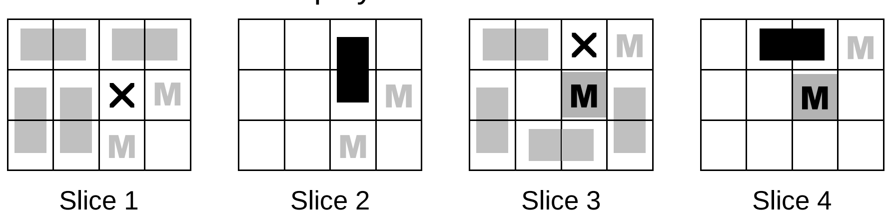


Diagram 4-22
After M’s fourth turn (indicated by the shaded square), the alliance plays moves above and below an available M move in Slice 3 that also block an available M move in Slice 1 (indicated by the two x’s)

In Diagram 4-22, the alliance cooperatively moves on Slices 2 and 4 after M’s fourth turn that not only blocks an available M move in Slice 3, but also blocks an M move in Slice 1.

Therefore, the alliance’s winning strategy for Case 1 is as follows.

- *If there are cooperative moves that can block more than one of M's available moves, then the alliance should play them.*
- *If that is not possible, then the alliance should play a pair of cooperative moves in their next turns that can block one of M's available moves.*
- *If cooperative blocking moves are not available, then both players should block different available moves for M in their next turns.*

If none of the above applies to an alliance member, then the member will need to iterate through his available moves set until he finds a valid move. A valid move in this case is one that would not prevent the other alliance member from playing a piece that would have blocked one of M's available moves.

Now that we have stated the alliance's winning strategy for the case where both alliance members cannot move on Slices 1 and 3 in their next turn, let's examine the case where only one of the alliance members cannot move on Slices 1 and 3 in his next turn.

Case 2: Only one alliance member can no longer play on Slices 1 and 3

The following diagram shows an example of what happens when one alliance member is no longer able to play on Slices 1 and 3.

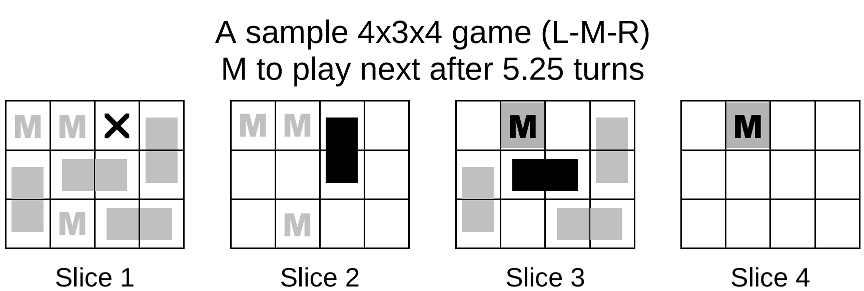


Diagram 4-23
After M's fourth turn (indicated by the shaded square), R plays on Slice 3. L is unable to play on Slice 1 or Slice 3, so he plays the first move that blocks M (indicated by the x).

In Diagram 4-23, after M makes his fourth turn (indicated by the shaded square), only one of the alliance members (R) is able to move on Slice 3. L would then need to play a piece that would block one of M's moves, which he does on Slice 2.

The one alliance member who cannot play on Slice 1 or Slice 3 should look for moves that can block one of M's available spaces using one piece. However, this alliance member must be careful if he happens to go before the other alliance member who can play on Slice 1 or 3. Here's why.

A sample 4x3x4 game (M-R-L)
R to play next after 4.5 turns

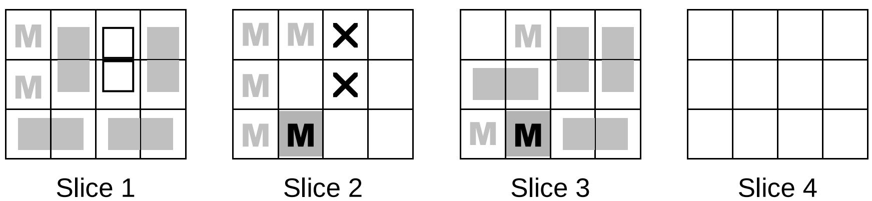


Diagram 4-24
After M's fifth turn (indicated by the shaded square), R should avoid playing on the squares marked with 'x' as those spaces will be blocked by L's next move on Slice 1 (indicated by the open squares).

Diagram 4-24 shows a sample 4x3x4 game with R going next. R could play on (2, 0, 1) or (0, 1, 1) to block one of M's remaining moves. However, L still has a move left on Slice 1 (indicated by the open squares) that would block two of M's spaces in Slice 2 (indicated by the x's). Therefore, R's move on Slice 2 would be a wasted move for the alliance since R could have blocked another M move that would not have been blocked by L's move.

A sample 4x3x4 game (M-R-L)
R to play next after 4.5 turns

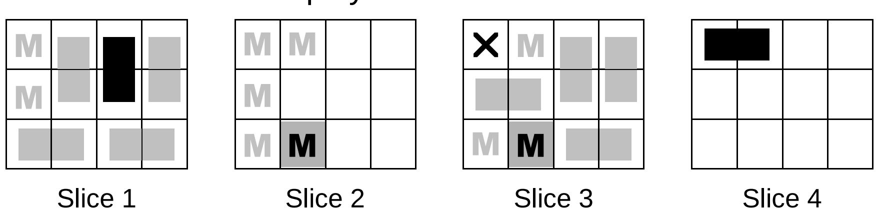


Diagram 4-25
After M's fifth turn (indicated by the shaded square), R played on Slice 4, blocking one of M's moves on Slice 3 (indicated by the x).
L played on Slice 1, removing the last of M's remaining moves.

In Diagram 4-25, R ignores playing on squares marked with x's in Diagram 4-24. Instead, R plays on the upper-left corner of Slice 4, blocking M from playing on the upper-left corner of Slice 3 (indicated by the x). L plays on Slice 1 after R, which removes the last of M's remaining moves, resulting in a loss for M.

The following summarizes the alliance's winning strategy for Case 2.

If only one alliance member cannot play on Slice 1 or Slice 3 in his next turn, he should play a move that can fully block an available M space using only one piece. However, if that member goes first, he must not block an available M space that will also be blocked by the other alliance member's next move on a slice that is in the Slices 1 & 3 pair.

This completes our analysis of the second part of the alliance's winning strategy.

4.3 – Results from Simulation using Alliance's Winning Strategy

Now that we have described the alliance's winning strategy to prevent M from winning a 4x3x4 game (where M plays on the 4 component), we conclude the proof of Proposition 4-1 by proving that the winning strategy works. We have done this by creating an application that simulates a three-player Domineering game and implemented the winning strategy described in the last two sections for two of the players. The application performs every possible move M can do throughout the game and chooses moves for L and R based on the winning strategy.

There is one thing to be aware about this application—it will stop analyzing a game if M is eliminated before L or R. The goal of the application is to check if there is a game where M wins against the alliance, as that would disprove our winning strategy and ultimately Proposition 4-1 (M is unable to force a win in a 4x3x4 game, where M plays on the 4 component, if the L & R alliance teams up against him). Therefore, if M is eliminated before L or R, then we do not care how that game ends.

Our application generated six text files from the 4x3x4 game simulation, one for each turn order (M-L-R, M-R-L, L-R-M, R-L-M, L-M-R, R-M-L). The following is an excerpt from one of those text files:

```
Program started at: 1284406810
Dimensions of Domineering board: 4x3x4
Turn order for all games in this simulation: left, right, middle

Game Number: 1 + (10000000 x 0)
      Winner: left/right
            left:    0,1,0  3,1,0  2,1,0  3,0,2  1,1,2  2,1,1
            right:   0,0,0  2,0,0  0,0,2  2,2,2  1,0,3  2,1,3
            middle:  1,1,0  1,2,0  0,1,1  0,2,1  2,0,1
            # of Remaining Moves: - left: 5 - right: 5 - middle: 0

.
.
.

Game Number: 114377 + (10000000 x 0)
      Winner: left/right
            left:    0,1,0  0,1,2  1,1,2  3,1,0  1,1,0  0,0,1
            right:   0,0,0  0,0,2  2,0,2  2,0,0  2,1,1  0,2,1
            middle:  3,2,2  3,1,2  2,2,2  2,1,2  2,2,0
            # of Remaining Moves: - left: 5 - right: 7 - middle: 0

Program finished at: 1284406831
```

Diagram 4-26
An excerpt from the output of the 4x3x4 L-R-M game simulation.

The first lines of the output indicate the time the program started, the dimensions of the board, and the turn order for all games simulated by the application. The simulation stops analyzing a game when either M wins or has no more available moves (regardless of whose turn it is). When that happens, it outputs how many times the simulation stopped analyzing a game (indicated in the Game Number line), whether M won or not, and what moves each player made for that specific game. It also writes the number of remaining moves left for each player. When the application finishes traversing through all possible games, it writes the time before it quits.

In each turn order output file, performing a search for “Winner: middle” yielded no results. This is how we determine that there are no games where M won when the alliance performed its winning strategy. If you were to take a random game from any of the output files and played the moves indicated in that game, it will show that the alliance performed their winning strategy to prevent M from winning.

Although our output text files indicate that M did not win any games, there are certain games where the last move recorded was made by M. To prove that M will not be able to force a win, the application performs an additional move for both alliance members to show that the alliance has at least one more move after M's final move. There are some games where one alliance member is eliminated before this additional move can be made. However, the other alliance member is still able to move after M's final move, which is a loss for M. The "# of remaining moves" line in the output reflects the additional moves made by the alliance after M's final turn.

This concludes the proof of Proposition 4-1.

The following two charts compare the performance of our simulation using the alliance's winning strategy against a simulation that perform a turn-order sub-tree exhaustive search of a 3D Domineering game.

4x3x4 3D Domineering Turn-Order Sub-tree Simulations Using Winning Strategy		
Turn Order	**Duration**	**Number of end nodes**
MLR	251 seconds	1,415,738
MRL	259 seconds	1,455,637
LMR	81 seconds	453,729
RML	83 seconds	458,325
LRM	21 seconds	114,377
RLM	21 seconds	116,839
Total:	**716 seconds**	**4,014,645**

Diagram 4-27

Exhaustive 3D Domineering Turn-Order Sub-tree Simulation		
Size & Turn Order	**Duration**	**Number of leaf nodes**
3x3x3 - MLR	Approx. 8 hours	1,034,224,512

Diagram 4-28

The chart in Diagram 4-27 demonstrates how long each of the six turn-order simulations lasted, how many end nodes they reached, and the total sum of each simulation's stats. Diagram 4-28 lists the performance of the simulation that traverses the game tree of a 3x3x3 M-L-R game.

In Chapter 1, we stated that in order to construct a game tree for a combinatorial game, every possible move that could be made at each position by all players must be listed, regardless of turn order. The simulation in Diagram 4-28 does not construct the entire game tree for the 3D Domineering game. Instead, it only considered moves at each position for whoever's turn it was, which is why it is labeled as a turn-order sub-tree simulation. This was done because calculating the complete game tree would have taken the simulation a long time to compute. The total number of leaf nodes in complete 3x3x3 game tree is significantly higher than the number of leaf nodes in its turn-order sub-tree counterpart. To demonstrate the size disparity between the complete tree and the turn-order sub-tree, we will estimate the total number of leaf nodes in a complete 3x3x3 game tree.

At each position of Diagram 4-28's 3x3x3 turn-order sub-tree, the simulation only considers the current player's options instead of all three players' options. In other words, 2/3 of a complete 3x3x3 game tree's branches are pruned at each level in the sub-tree simulation. We can approximate the total number (K) of end nodes of Diagram 4-28's 3x3x3 simulation with the following equation:

$$K \cong (2/3)^n \bullet J$$

where J represents the total number of leaf nodes in a complete 3x3x3 game tree, and n represents the height of the complete game tree. There are 27 spaces in a 3x3x3 game, and if each player tried to pack as many pieces as possible in the board, there would be a total of 27/2 = 13 moves made by all three players. Thus, we can presume that an upper bound for the height of the 3x3x3 game tree is 13. If we were to rewrite the equation to solve for J, we would get:

$$J \cong (3/2)^{13} \bullet K \approx 194.61 \bullet K$$

Replacing K with the number of leaf nodes defined in Diagram 4-28 for a 3x3x3 turn-order sub-tree, the total number of leaf nodes in a complete 3x3x3 game tree is approximately 201,280,264,483. This is significantly more than the leaf nodes listed for the 3x3x3 turn-order sub-tree in Diagram 4-28. We can make the case that the total number of leaf nodes for a complete 4x3x4 game tree would be significantly higher than the number of leaf nodes in a complete 3x3x3 game. We make this point because the total time and number of end nodes from our simulation would clearly be significantly less than an analysis of a complete 4x3x4 game.

Using our application, we have demonstrated that regardless of turn-order, M cannot force a win in a 4x3x4 Domineering game, where M plays on the second 4 component, if the alliance teams up against M using the winning strategy we outlined in this chapter.

We have demonstrated in the last two chapters that M cannot force a win in a 4x4x3 game if L and R collude against him, regardless of turn order or whether M plays on the 3 component or the second 4 component. If M cannot force a win if L and R team up against him, it also means that L cannot force a win if R and M team up against him and R cannot force a win if L and M team up against him. Therefore, no player can force a win in a 4x4x3 game, regardless of board orientation or turn order. **Thus, we have proven that a 4x4x3 game is solved to be in the outcome class $\mathcal{Q}$.**

5.0 – Future Work

Completely solving a 4x4x3 Domineering game using the strategies outlined in the previous chapters opens the possibility of solving 3D Domineering boards of larger sizes without having to do a complete traversal of their game trees. We previously determined that the alliance's optimal strategy to prevent M from winning 3x3x3 and 4x4x3 (where M plays on the 3 component) games is to cover as much of the middle slice as possible. One consideration for future work is to investigate if there is a limit to how large a game of m x n x 3 (where $m > 4$, $n > 4$, and M plays on the 3 component) can be where blocking the middle slice remains an effective strategy for the alliance to block M.

Another consideration for future work is to test if having the alliance play on alternating slices is an effective strategy to prevent M from winning regardless of how many slices M can play on. We found that when M plays on four slices in a 4x3x4 game, the strategy to block M is to play on alternating slices (Slices 1 & 3). If M played on five slices (for example, 5x5x5), can the alliance prevent M from winning by playing on Slices 2 & 4? And if M played on six slices (for example, 6x6x6), can the alliance prevent M from winning if they played on Slices 1, 3, and 5? What if the board size was skewed to M's favor, such as a 3x3x8 board, with M playing on the 8 component? Would playing on alternate slices still be enough for the alliance to prevent M from winning? How big does the board have to skew in M's favor before M can force a win despite the other two players teaming up against him? These are questions that can be explored with further research.

Yet another consideration for future work is improving how the simulation application chooses moves for the alliance. For the 4x3x4 game, the application either chose hard-coded moves or the first available moves that fulfilled certain criteria (blocked M's move, did not block other alliance member, etc). An improvement to the application would consider ranking available moves for both alliance members and choosing the one that would most help the alliance defeat M. The process of ranking moves would prolong the amount of time it takes for the application to simulate a three-player game. However, its implementation could be applied to larger boards without having to write specialized code for the game similarly to the ones written for the 4x3x4 game.

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